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# THE STRUCTURE OF IDEALS IN A POLYNOMIAL SEMIRING IN SEVERAL VARIABLES

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## 1. Introduction

In recent literature we have examined the structure of monic, monic free and $k$-ideals in a polynomial semiring. In this paper we extend this examination into polynomial semirings in several variables. We do this by investigating a class of polynomials, called saturated polynomials, that can be grouped together naturally to form ideals. It happens that these ideals form a basic structure for arbitrary ideals in a polynomial semiring in several variables. We develop a structure theorem for ideals and apply this theorem to monic, monic free and $k$-ideals.

Let $S$ be a semiring with an identity and $x_1, x_2, \cdots, x_n$ be indeterminates which commute with each other and with each element of $S$. Then $S[x_1, x_2, \cdots, x_n]$ is a semiring and a typical element of $S[x_1, x_2\cdots, x_n]$ is of the form $\Sigma a_{i_1 i_2\cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ where $i_1, i_2, \cdots, i_n$ are nonnegative integers and $a_{i_1 i_2 \cdots i_n} \in S$. If we let $\alpha = \{i_1, i_2, \cdots, i_n\}$, $\Phi = x_1 x_2 \cdots x_n$, $a_\alpha = a_{i_1 i_2 \cdots i_n}$ and $\Phi^\alpha = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ then we can denote $\Sigma a_{i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ by $\Sigma a_\alpha \Phi^\alpha$. With this notation we call an ideal $M$ in $S[x_1, x_2, \cdots, x_n]$ a *monic* ideal if $\Sigma a_\alpha \Phi^\alpha \in M$ implies $a_\alpha \Phi^\alpha \in M$ for each $\alpha$ and an ideal $F$ is called *monic free* if $M$ is a monic ideal such that $M \subseteq F$, then $M = \{0\}$. An ideal $B$ in a semiring $S$ is called a *k-ideal* in $S$ if $a \in B$, $c \in S$ and $a + c \in B$ imply $c \in B$. Finally we call a semiring $S$ a *strict* semiring if $a, b \in S$ and $a + b \in S$ imply $a = b = 0$. For this paper all semirings will be commutative and strict.

## 2. Saturated ideals

Consider an ideal $A$ in a polynomial semiring $S[x_1, x_2, \cdots, x_n]$. We want to find a way to partition the elements of $A$ in some natural arrangement. We will call a polynomial $f = \Sigma a_\alpha \Phi^\alpha$ in $S[x_1, x_2, \cdots, x_n]$ *saturated* in $S[x_1, x_2, \cdots, x_n]$ if either $f = 0$ or each $x_i \in X = \{x_1, x_2, \cdots, x_n\}$ appears in some nonzero term of $f$. For example, the polynomial $f = 3x + 6xy + 7yz + 5$ is saturated in $Z[x, y, z]$ but not in $Z[w, x, y, z]$. If $S$ is a strict semiring and $f, g$ are saturated in $S[x_1, x_2, \cdots, x_n]$, then it is clear that $f+g$, $fg$ and any multiple of $f$ are all saturated in $S[x_1, x_2, \cdots, x_n]$, i.e., the set $S_X$ of all saturated polynomials in $S[x_1, x_2, \cdots, x_n]$ is an ideal. Consequently, if $A$ is an ideal in $S[x_1, x_2, \cdots, x_n]$, then $A_X = A \cap S_X$ is an ideal in $S[x_1, x_2, \cdots, x_n]$. Let $\tau = \{x_1', x_2', \cdots, x_t'\}$ be a subset of $X$. Then $S_\tau$ and $A_\tau = A \cap S_\tau$ are ideals in $S[x_1', x_2', \cdots, x_t']$. If $\tau = \phi$, define $A_\tau = A \cap S$. Note that if $\tau \neq \delta$, then $A_\tau \cap A_\delta = \{0\}$. Thus every $f = \sum a_\alpha \Phi^\alpha \in A$ belongs to one and only one $A_\tau$. Consequently, any ideal $A$ in $S[x_1, x_2, \cdots, x_n]$ can be decomposed into a finite number of saturated subideals, i.e., $A = \bigcup_{\tau \subseteq X} A_\tau$ where $A_\tau$ is saturated in $S[x_1', x_2', \cdots, x_t']$ and $A_\tau \cap A_\delta = \{0\}$ if $\tau \neq \delta$. Now $A_\tau$ is an ideal in $S[x_1', x_2', \cdots, x_t']$ but not in $S[x_1, x_2, \cdots, x_n]$. However, $A_\tau$ generates an ideal $(A_\tau)$ in $S[x_1, x_2, \cdots, x_n]$. This proves the following structure theorem.

THEOREM 2.1. *If $A$ is an ideal in $S[x_1, x_2, \cdots, x_n]$, then $A = \sum_{\tau \subseteq X} (A_\tau)$, where $A_\tau \cap A_\delta = \{0\}$ for $\tau \neq \delta$.*

This decomposition in a sense partitions $A$ into a sum of simpler ideals. The question now is how useful is this decomposition? We have one use in the following theorem.

THEOREM 2.2. *An ideal $A$ in $S[x_1, x_2, \cdots, x_n]$ is monic free if and only if $(A_\tau)$ is monic free for each $\tau \subseteq X$.*

PROOF. Suppose $A$ is a monic free ideal in $S[x_1, x_2, \cdots, x_n]$ and $M$ is a monic ideal such that $M \subseteq (A_\tau)$. From $M \subseteq (A_\tau) \subseteq A$ it follows that $M = \{0\}$ and $(A_\tau)$ is monic free. Next suppose $(A_\tau)$ is monic free for each $\tau \subseteq X$ and $M$ is monic such that $M \subseteq A$. Since $M$ is monic we must have $a_\alpha \Phi^\alpha \in M$ for some $a_\alpha \in S$. Now $A = \bigcup_{\tau \subseteq X} A_\tau$ and it follows that $a_\alpha \Phi^\alpha \in A_\tau$ for some $\tau$ and consequently, $a_\alpha \Phi^\alpha \in (A_\tau)$. Since $(A_\tau)$ is monic free we must have $a_\alpha = 0$ and consequently, $M = \{0\}$. Therefore $A$ is monic free.

### 3. The structure of monic and $k$-ideals

The decomposition of an ideal $A$ in $S[x_1, x_2, \cdots, x_n]$ given in theorem 2.1 is useful in dealing with monic free ideals but not useful in dealing with monic ideals or $k$-ideals. The elements in these ideals sometimes split into smaller pieces. For if $f+g\in(A_\tau)$ we can make no claim concerning whether or not $f$ or $g\in(A_\tau)$ using the fact that $A_\tau$ is saturated. Hence we need to alter our decomposition a bit to make it more useful. To do this let

$$\tau=\{x_1{}',x_2{}',\cdots,x_t{}'\}\subseteq X \text{ and } P_\tau=\sum_{\delta\subseteq\tau}(A_\delta).$$

Now $P_\tau$ is an ideal in $S[x_1,x_2,\cdots,x_n]$ and $A_\tau\subseteq P_\tau$.

THEOREM 3.1. *If $A$ is an ideal in $S[x_1,x_2,\cdots,x_n]$ and $\tau\subseteq X$, then*

$$A=P_\tau+\sum_{\lambda\not\subseteq\tau}A_\lambda.$$

PROOF. It follows from theorem 2.1 that $A=\sum_{\tau\subseteq X}(A_\tau)$. But

$$A=\sum_{\tau\subseteq X}(A_\tau)=\sum_{\delta\subseteq\tau}(A_\delta)+\sum_{\lambda\not\subseteq\tau}A_\lambda=P_\tau+\sum_{\lambda\not\subseteq\tau}A_\lambda.$$

This theorem in a sense factors the ideal $A$ into two ideals $P_\tau$ and $\sum_{\lambda\not\subseteq\tau}A_\tau$. This factorization is not unique since $\tau\subseteq X$ is arbitrary. Thus we get a decomposition for $A$ for each $\tau\subseteq X$.

THEOREM 3.2. *If $A$ is an ideal in $S[x_1,x_2,\cdots,x_n]$ and $\tau,\lambda\in X$, then $P_\tau\cap P_\lambda=P_{\tau\cap\lambda}$.*

PROOF. Let $f_\mu$ be a polynomial in $P_\tau\cap P_\lambda$. Then $f_\mu\in A_\mu$, $f_\mu\in P_\tau$ and $f_\mu\in P_\lambda$ and it follows that $\mu\subseteq\tau$ and $\mu\subseteq\lambda$. Consequently, $\mu\subseteq\tau\cap\lambda$ and thus $f_\mu\in P_{\tau\cap\lambda}$. Hence $P_\tau\cap P_\lambda\subseteq P_{\tau\cap\lambda}$. Reversing the steps will show that $P_{\tau\cap\lambda}\subseteq P_\tau\cap P_\lambda$ and consequently, that $P_{\tau\cap\lambda}=P_\tau\cap P_\lambda$.

Now if $f+g$ is a saturated polynomial in $P_\tau$ we know that $f\in A_{\lambda_1}$ and $g\in A_{\lambda_2}$ for some $\lambda_1,\lambda_2\subseteq\tau$. Thus the structure of $P_\tau$ allows us to know that $f$ or $g\in P_\lambda$. This property of $P_\lambda$ allows us to prove the following.

THEOREM 3.3. *An ideal $A$ in $S[x_1,x_2,\cdots,x_n]$ is monic if and only if $P_\tau$ is monic for each $\tau\subseteq X$.*

PROOF. Suppose $A$ is a monic ideal and $f=\Sigma a_\alpha\Phi^\alpha\in P_\tau$. Now $a_\alpha\Phi^\alpha\in A$ since $A$ is monic. But $A=\bigcup_{\tau\subseteq X}A_\delta$ and it follows that $a_\alpha\Phi^\alpha\in A_\lambda$ for some $\lambda\subseteq X$. But $f\in P_\tau$ implies that $\lambda\subseteq\tau$. Hence $a_\alpha\Phi^\alpha\in A_\lambda\subseteq P_\tau=\sum_{\lambda\subseteq\tau}(A_\lambda)$ and it follows that $P_\tau$ is a monic ideal. Conversely, suppose $P_\tau$ is monic for each $\tau\subseteq X$. Since $f\in A$ implies that $f\in A_\tau\subseteq P_\tau$ for some $\tau\subseteq X$, it follows that $A$ must be monic also.

THEOREM 3.4. *An ideal $A$ in $S[x_1, x_2, \cdots, x_n]$ is a $k$-ideal if and only if $P_\tau$ is a $k$-ideal for each $\tau\subseteq X$.*

PROOF. Suppose $A$ is a $k$-ideal and $f, f+g\in P_\tau$. Since $A$ is a $k$-ideal and $P_\tau\subseteq A$ it follows that $g\in A$. Now $A=\bigcup_{\delta\subseteq X} A_\delta$ and it follows that $g\in A_\lambda$ for some $\lambda\subseteq X$. But $f+g\in P_\tau$ implies that $\lambda\subseteq\tau$. Hence $g\in A_\lambda\subseteq P_\tau=\sum_{\lambda\subseteq\tau}(A_\lambda)$ and consequently, $P_\tau$ is a $k$-ideal. Conversely suppose that $P_\tau$ is a $k$-ideal for each $\tau\subseteq X$ and $f, f+g\in A$. Now $f\in A_{\lambda_1}$ and $f+g\in A_{\lambda_2}$ for some $\lambda_1, \lambda_2\subseteq X$. Let $\tau=\lambda_1\cup\lambda_2$. Then $f, f+g\in P_\tau$ and since $P_\tau$ is a $k$-ideal it follows that $g\in P\subseteq A$ and $A$ is a $k$-ideal.

It is interesting to note that an ideal $A$ can be written $A=P_\tau+\sum_{\lambda\not\subseteq\tau}(A_\lambda)$ for each $\tau\subseteq X$. But $P_\tau$ is an ideal in $S[x_1, x_2, \cdots, x_n]$ and we must be able to write $P_\tau=P_\delta+\sum_{\lambda\not\subseteq\delta}(A_\lambda)$. Thus $\delta\subseteq\tau\subseteq X$ and $P_\delta\subseteq P_\tau\subseteq A$. Consequently, if $X\supseteq\tau_1\supseteq\tau_2\supseteq\cdots\supseteq\tau_n$ is a descending chain of subsets of $X$, then $A\supseteq P_{\tau_1}\supseteq P_{\tau_2}\supseteq\cdots\supseteq P_{\tau_n}$ is a descending chain of ideals. This fact and theorem 3.4 proves the following.

THEOREM 3.5. *If $A$ is an ideal in $S[x_1, x_2, \cdots, x_n]$ and $X\supseteq\tau_1\supseteq\tau_2\supseteq\cdots\supseteq\tau_n$ is a descending sequence of subsets of $X$, then $A\supseteq P_{\tau_1}\supseteq P_{\tau_2}\supseteq\cdots\supseteq P_{\tau_n}$. Further, if $A$ is a $k$-ideal, then each $P_{\tau_i}$ is a $k$-ideal.*

The ideals $(A_\tau)$ and $P_\tau$ may look similar but they are very different. $(A_\tau)$ may be monic free while $P_\tau$ is monic and $(A_\tau)$ be a $k$-ideal while $P_\tau$ is not.

### 4. Weak $k$-ideals and $k$-closures

It is well known that every ideal in a semiring is not a $k$-ideal but every ideal is contained in a $k$-ideal. Let $A$ be an ideal in a semiring $S$. The ideal

$$\overline{A}_k=\cap\{B\mid B \text{ is a } k\text{-ideal and } B\supseteq A\}$$

is called the *$k$-closure* of $A$. It is clear that $\overline{A}_k$ is the "smallest" $k$-ideal containing $A$. We call an ideal $A$ in $S[x_1, x_2, \cdots, x_n]$ a *weak $k$-ideal* if there is an integer $n$ such that $A$ is a $k$-ideal with respect to all polynomials in $A$ with degree less than or equal to $n$. The largest such integer is called the *$k$-degree* of $A$. If no such integer exists then $A$ is said to have $k$-degree $\infty$. Now if $A$ is an ideal in $S[x_1, x_2, \cdots, x_n]$ and $n$ is a fixed integer let

$$A_{f_n}=\{f|f\in A \text{ and degree } f\leq n\}.$$

The ideal

$\bar{A}_{k_n}=\cap\{B|B$ is a weak $k$-ideal with $k$-degree at least $n$ and $A_{f_n}\subseteq B\}$ will be called the weak $k$-closure of $A$.

We now apply these concepts to saturated polynomials. If $A$ is an ideal in $S[x_1, x_2, \cdots, x_n]$ let

$$A_{\tau f_n}=A_\tau\cap A_{f_n}.$$

Thus $A_{\tau f_n}$ is the set of all polynomials in $A$ saturated in $S[x_1', x_2', \cdots, x_t']$ with degree less than or equal to $n$. It is clear that $A_{f_n}=\bigcup_{\tau\subseteq X} A_{\tau f_n}$. Now we have $A=\bigcup_n A_{f_n}$ and from section 2 we have $A=\bigcup_{\tau\subseteq X} A_\tau$. Hence

$$A=\bigcup_n A_{f_n}=\bigcup_n\{\bigcup_{\tau\subseteq X} A_{\tau f_n}\}.$$

We will denote the weak $k$-closure of the ideal $(A_\tau)$ by $\overline{(A_\tau)}_{k_n}$ and the $k$-closure of $A_\tau$ by $\overline{(A_\tau)}_k$.

THEOREM 4.1. *If $A$ is an ideal in $S[x_1, x_2, \cdots, x_n]$ then* $\bar{A}_k=\sum_{\tau\subseteq X}\overline{(A_\tau)}_k$.

PROOF. From theorem 2.1 we have $A=\sum_{\tau\subseteq X}(A_\tau)$ where $A_\tau\cap A_\delta=\{0\}$ for $\tau\neq\delta$. Since $\bar{A}_k$ is an ideal in $S[x_1, x_2, \cdots, x_n]$ we have $\bar{A}_k=\sum_{\tau\subseteq X}(\bar{A}_k)_\tau$. Now $(A_\tau)$ is an ideal in $S[x_1, x_2, \cdots, x_n]$ and $\overline{(A_\tau)}_k$ is the smallest $k$-ideal containing $(A_\tau)$. Also $(\bar{A}_k)_\tau$ is a $k$-ideal in $S[x_1, x_2, \cdots, x_n]$ containing $(A_\tau)$ and $\overline{(A_\tau)}_k$. Since $\bar{A}_k$ is the smallest $k$-ideal containing $A$ it follows that $(\bar{A}_k)_\tau$ is the smallest $k$-ideal containing $(A_\tau)$. Consequently, $(\bar{A}_k)_\tau=\overline{(A_\tau)}_k$ and it follows that $\Sigma(\bar{A}_k)_\tau=\Sigma\overline{(A_\tau)}_k$ and hence $\bar{A}_k=\sum_{\tau\subseteq X}\overline{(A_\tau)}_k$.

COROLLARY 4.2. *If $A$ is an ideal in $S[x_1, x_2, \cdots, x_n]$ then* $\bar{A}_k=\overline{(P_\tau)}_k+\sum_{\lambda\not\subseteq\tau}\overline{(A_\tau)}_k$.

PROOF. Theorem 4.1 assures that

$$\bar{A}_k=\sum_{\tau\not\subseteq X}\overline{(A_\tau)}_k=\sum_{\delta\subseteq\tau}\overline{(A_\delta)}_k+\sum_{\lambda\not\subseteq\tau}\overline{(A_\lambda)}_k=\overline{(P_\tau)}_k+\sum_{\lambda\not\subseteq\tau}\overline{(A_\lambda)}_k.$$

## 5. Example

Let $A=(2, x^n+2, y^m+2, z^t)$, $n>m>t>1$, be an ideal in $Z^+[x,y,z]$. Now $A$ is a weak $k$-ideal of degree $(m-1)$. To see this we check the basis elements of $A$ and note that $y^m+2$, $2\in A$ but $y^m\notin A$. Thus $A$ is not a $k$-ideal. Now any polynomial in $y$ of degree $(m-1)$ or less is of the form $2\sum a_i y^i$ and $A$ is certainly a $k$-ideal with respect to these polynomials since $2a_i y^i \in A$. Similarly, $A$ is a $k$-ideal with respect to all polynomials in $x$ of degree $(n-1)$ or less, and $A$ is a $k$-ideal with respect to all polynomials in $z$. Since $n>m$, $(n-1)>(m-1)$ and it follows that $A$ is a weak $k$-ideal with $k$-degree $(m-1)$. Now let us partition $A$ into saturated polynomials. Let $\tau_0=\phi$, $\tau_1=\{x\}$, $\tau_2=\{y\}$, $\tau_3=\{z\}$, $\tau_4=\{x,y\}$, $\tau_5=\{x,z\}$, $\tau_6=\{y,z\}$, $\tau_7=\{x,y,z\}=X$.

Then the saturated ideals are

$$A_{\tau_0}=(2),\ A_{\tau_1}=(x^n+2)\subseteq Z^+[x],\ A_{\tau_2}=(y^m+2)\subseteq Z[y],$$
$$A_{\tau_3}=(z^t)\subseteq Z^+[z],\ A_{\tau_4}=(x^n+y)+(y^m+2)\subseteq Z^+[y,z],$$
$$A_{\tau_5}=(x^n+2)+(z^t)\subseteq Z^+[x,z],\ A_{\tau_6}=(y^m+2)+(z^t)\subseteq Z[y,z],$$
$$A_{\tau_7}=(x^n+2)+(y^m+2)+(z^t)\subseteq Z^+[x,y,z].$$

Now

$P_{\tau_0}=(A_{\tau_0})$ and $\bar{P}_{\tau_0 k}=(A_{\tau_0})$,

$P_{\tau_1}=(A_{\tau_1})+(A_{\tau_0})=(x^n+2)+(2)$ and $\bar{P}_{\tau_1 k}=(x^n,2)$

$P_{\tau_2}=(A_{\tau_2})+(A_{\tau_0})=(y^m+2)+(2)$ and $\bar{P}_{\tau_2 k}=(y^m,2)$

$P_{\tau_3}=(A_{\tau_3})+(A_{\tau_0})=(z^t)+(2)$ and $\bar{P}_{\tau_3 k}=P_{\tau_3}$

$P_{\tau_4}=(A_{\tau_4})+(A_{\tau_1})+(A_{\tau_2})+(A_{\tau_0})=(x^n+2)+(y^m+2)+(2)$ and $\bar{P}_{\tau_4 k}=(x^n,y^m,2)$

$P_{\tau_5}=(A_{\tau_5})+(A_{\tau_3})+(A_{\tau_1})+(A_{\tau_0})=(x^n+2)+(z^t)+(2)$ and $\bar{P}_{\tau_5 k}=(x^n,z^t,2)$

$P_{\tau_6}=(A_{\tau_6})+(A_{\tau_3})+(A_{\tau_2})+(A_{\tau_0})=(y^m+2)+(z^t)+(2)$ and $\bar{P}_{\tau_6 k}=(y^m,z^t,2)$

$P_{\tau_7}=A$ and $\bar{P}_{\tau_7 k}=\bar{A}_k=(x^n,y^m,z^t,2)$.

A few observations.

1. The $k$-degree of $P_{\tau_2}, P_{\tau_4}, P_{\tau_6}$ and $P_{\tau_7}$ is $(m-1)$, while the $k$-degree of $P_{\tau_1}$ and $P_{\tau_5}$ is $(n-1)$, and the $k$-degree of $P_{\tau_3}$ is $\infty$.
2. Each $A_\tau$ is a $k$-ideal and has $k$-degree $\infty$.
3. It is clear that for each $\tau\subseteq X$

$$k\text{-degree } A\leq k\text{-degree } P_\tau \leq k\text{-degree } A_\tau.$$

4. $X\supseteq\tau_5\supseteq\tau_3\supseteq\tau_0$ and it follows that $A\supseteq P_{\tau_5}\supseteq P_{\tau_3}\supseteq P_{\tau_0}$ is a descending chain of ideals. Also $\bar{A}_k\supseteq\bar{P}_{\tau_5 k}\supseteq\bar{P}_{\tau_3 k}\supseteq\bar{P}_{\tau_0 k}$ is a descending chain of $k$-ideals.

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