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CERTAIN CONTACT CR-SUBMANIFOLDS WITH FLAT NORMAL CONNECTION OF A SASAKIAN SPACE FORM

(Dedicated to professor Kee Ahn Lee on his sixtieth birthday)

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0. Introduction

Recently the contact *CR*-submanifolds of a Sasakian manifold have been defined and studied by Yano and Kon [6] and are now being studied by many authors [2, 3, 4].

The main purpose of the present paper is to study compact contact *CR*-submanifolds of an odd-dimensional unit sphere.

In 1, we first of all recall some well known results on submanifolds of a Sasakian manifold. In 2, we introduce some lemmas on contact *CR*-submanifolds of a Sasakian manifold.

Section 3 is devoted to study compact contact *CR*-submanifolds of an odd-dimensional unit sphere.

1. Submanifolds of Sasakian manifolds

Let N be a $(2m+1)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U; y^h\}$ and with structure tensors (F_i^h, g_{ji}, v^h). Then we have

$$(1.1)\qquad F_i^h F_h^j = -\delta_i^j + v_i v^j,\ v_i F_j^i = 0,\ F_i^j v^i = 0,\ v_i v^i = 1,\ g_{hk} F_j^h F_i^k = g_{ji} - v_j v_i,$$

v_i being the associated 1-form of v^i, where here and in the sequel, the indices h, i, j, k run over the range $\{1, 2, \cdots, 2m+1\}$. Denoting by ∇_i the operator of covariant differentiation with respect to g_{ji}, we also have

$$(1.2)\qquad \nabla_j F_i^h = -g_{ji} v^h + \delta_j^h v_i,\ \nabla_j v^i = F_j^i.$$

Let M be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; x^a\}$ and isometrically immersed in N by the immersion $i: M \to N$. We identify $i(M)$ with M itself and represent the immersion locally by $y^h = y^h(x^a)$. Throughout this paper, the indices a, b, c, d, e run over the range $\{1, 2, \cdots, n+1\}$ and we assume that the submanifold M of N is tangent to the structure vector field v^h. If we put $B_a^i = \partial_a y^i$, $\partial_a = \partial/\partial x^a$, then B_a^i are $(n+1)$-linearly independent vectors of M tangent to M. We denote by

C^i_x, $p(=2m-n)$ mutually orthogonal unit vectors normal to M. Since the immersion is isometric, we have

$$g_{cb}=g_{ji}B^j_cB^i_b,\ g_{xy}=g_{ji}C^j_xC^i_y,\ g_{ji}B^j_aC^i_x=0, \tag{1.3}$$

g_{cb} and g_{xy} being the induced metric tensor of M and that of the normal bundle of M respectively, where here and in the sequel the indices x, y, z, u, v, w run over the range $\{1, 2, \cdots, 2m-n=p\}$.

Denoting by ∇_a the operator of van der Waerden-Borotolotti covariant differentiation with respect to g_{ba}, we obtain equations of the Gauss and Weingarten for M

$$\nabla_cB^h_b=h_{cb}{}^xC^h_x,\ \nabla_cC^h_x=-h_{cx}^eB^h_e \tag{1.4}$$

respectively, where $h_{cb}{}^x$ are second fundamental tensors with respect to the normals C^h_x, $h^a_{cx}=h_{cb}{}^yg^{ba}g_{xy}$, and $(g^{ba})^{-1}=(g_{ba})$.

The mean curvature vector $h^x=g^{cb}h_{cb}{}^x$ of M is said to be parallel (in the normal bundle of M) if $\nabla_ah^x=0$.

If the ambient manifold N is a $(2m+1)$-dimensional unit sphere $S^{2m+1}(1)$, then the equations of Gauss, Codazzi and Ricci for M are given respectively by

$$K_{dcb}{}^a=\delta^a_dg_{cb}-\delta^a_cg_{db}+h^a_{dx}h_{cb}{}^x-h^a_{cx}h_{db}{}^x, \tag{1.5}$$

$$\nabla_ch_{ba}{}^x-\nabla_bh_{ca}{}^x=0, \tag{1.6}$$

$$K_{cby}{}^x=h_{ce}{}^xh_{by}{}^e-h_{be}{}^xh_{cy}{}^e, \tag{1.7}$$

where $K_{dcb}{}^a$ and $K_{dcy}{}^x$ are the the curvature tensor of M and that of the normal bundle of M respectively.

From now on we consider the transforms of B^i_a and C^i_x by F. Then we can put in each coordinate neighborhood

$$F^h_iB^i_a=f^b_aB^h_b+f^x_aC^h_x,\ F^h_iC^i_x=-f^a_xB^h_a+f^y_xC^h_y, \tag{1.8}$$

where f^b_a is a tensor field of type $(1,1)$, f^x_a a normal bundle valued 1-form, $f^a_x=f^y_bg^{ba}g_{yx}$, and f^y_x normal bundle valued function. Since the structure vector v^i is tangent to M, we can also put

$$v^i=v^aB^i_a \tag{1.9}$$

for a vector field v^a on M.

As in usual way, applying F to (1.8) and (1.9) gives

(1.10)
$$f_c^e f_e^a = -\delta_c^a + v_c v^a + f_c^x f_x^a,\ f_c^e f_e^x + f_c^y f_y^x = 0,$$
$$f_x^y f_y^z = -\delta_x^z + f_x^e f_e^z,\ v_a v^a = 1,\ v^a f_a^b = 0,\ v^a f_a^x = 0,$$
$$g_{de} f_c^d f_b^e = g_{cb} - f_c^x f_{bx} - v_c v_b,$$
where $v^b = v_a g^{ab}$ and $f_{bx} = f_b^y g_{yx}$. Moreover, putting $f_{cb} = f_c^a g_{ab}$, $f_{xc} = f_x^b g_{bc}$ and $f_{xy} = f_x^z g_{zy}$, we can see that

(1.11)
$$f_{cb} = -f_{bc},\ f_{xc} = f_{cx},\ f_{xy} = -f_{yx}.$$
Differentiating (1.8) and (1.9) covariantly along M and making use of (1.2), (1.4) and these equations, we also easily find that

(1.12)
$$\nabla_c f_b^a = \delta_c^a v_b - g_{cb} v^a + h_{cx}^a f_b^x - h_{cb}^x f_x^a,$$
(1.13)
$$\nabla_c f_b^x = h_{cb}^y f_y^x - h_{ce}^x f_b^e,\ \nabla_c f_x^a = h_{cx}^e f_e^a - h_{cy}^a f_x^y,$$
(1.14)
$$\nabla_c f_y^x = h_{ce}^x f_y^e - h_{cy}^e f_e^x,$$
(1.15)
$$\nabla_c v^a = f_c^a,$$
(1.16)
$$h_{ce}^x v^e = f_c^x.$$

2. Contact CR-submanifolds of Sasakian manifolds

Let M be a submanifold isometrically immersed in a Sasakian manifold N tangent to the structure vector v. Then M is called a *contact CR-submanifold* of N if there exists a differentiable distribution $D: p \to D_p T_p(M)$ on M satisfying the following conditions:

(I) D is invariant with respect to F, that is, $FD_p \subset D_p$ for each point p in M, and

(II) the complementary orthogonal distribution $D^\perp: p \to D_p^\perp \subset T_p(M)$ is anti-invariant with respect to F, that is, $FD_p^\perp \subset T_p^\perp(M)$ for each point $p \in M$.

For a contact CR-submanifold M of a Sasakian manifold N, the following theorem is well known:

THEOREM A([4], [6], [7], [8]). *In order for a submanifold M to be a contact CR-submanifold, it is necessary and sufficient that*

(2.1)
$$f_c^e f_e^x = 0 \ (\textit{equivalently}\ f_a^x f_x^y = 0).$$
In such a case, f_c^a and f_x^y are f-structure in M and that in the normal bundle of M respectively.

REMARK. Let M be a contact CR-submanifold of a Sasakian manifold N. If dim $D=0$, then M is an anti-invariant submanifold of N, and if dim $D^{\perp}=0$, then M is an invariant submanifold of N. If $FD^{\perp}=T(M)^{\perp}$, then M is a generic submanifold of N.

In the following we state two theorems concerning contact-submanifolds of $S^{2m+1}(1)$.

THEOREM B([6], [7]). *Let M be an $(n+1)$-dimensional complete contact CR-submanifold of $S^{2m+1}(1)$ with flat normal connection. If the mean curvature vector of M is parallel, and if*

$$h_{ce}^{x}f_{b}^{e}+h_{be}^{x}f_{c}^{e}=0$$

at every point in M, then M is an S^{n+1} or

$$S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k),\ n+1=\sum_{i=1}^{k}m_i,\ 2\leq k\leq n+1,\ \sum_{i=1}^{k}r_i^2=1$$

in some S^{n+1+q}, where $m_1,\cdots m_k$ are odd numbers and $q=\dim D^{\perp}$.

THEOREM C([8]). *Let M be an $(n+1)$-dimensional compact contact CR-submanifold with flat normal connection of $S^{2m+1}(1)$. If the mean curvature vector of M is parallel, and if*

$$h_{ce}^{x}f_{b}^{e}-h_{be}^{x}f_{c}^{e}=0$$

at every point in M, then M is a $S^{n+1}(1)$ or

$$S^1(r_1)\times\cdots\times S^1(r_{n+1})\ \text{in a}\ S^{2n+1}(1)\ \text{in}\ S^{2m+1}(1)$$

where $\sum_{i=1}^{n+1}r_i^2=1$ and $n+1$ is an odd number.

THEOREM D([5]). *Let M be an $(n+1)$-dimensional compact anti-invariant submanifold with parallel mean curvature vector and with flat normal connection of $S^{2m+1}(1)$. If the f-structure in the normal bundle is parallel, then M is*

$$S^1(r_1)\times\cdots\times S^1(r_{n+1})\ \text{in a}\ S^{2n+1}(1)\ \text{in}\ S^{2m+1}(1),$$

where $\sum_{i=1}^{n+1}r_i^2=1$.

3. Contact CR-submanifolds of $S^{2m+1}(1)$

We first prepare the following lemmas:

LEMMA 3.1 ([4], [6]). *Let M be a contact CR-submanifold with flat normal connection of $S^{2m+1}(1)$. Then the f-structure in the normal bundle of M is parallel, that is,* (3.1) $$\nabla_c f_y^x = h_{ce}^{\ \ x} f_y^e - h_{cy}^e f_e^x = 0.$$

LEMMA 3.2 ([4], [6]). *Let M be a contact CR-submanifold with flat normal connection of $S^{2m+1}(1)$. Then we have*

(3.2) $$h_{cb}^{\ \ x} f_x^y = 0.$$

LEMMA 3.3 ([8]). *Let M be an $(n+1)$-dimensional contact CR-submanifold with flat normal connection of $S^{2n+1}(1)$. Then $A = f_{cx} f^{cx}$ is constant on M, and if $A = n$, then M is anti-invariant submanifold of $S^{2m+1}(1)$.*

Let M be an $(n+1)$-dimensional contact CR-submanifold with flat normal connection of $S^{2m+1}(1)$.

Suppose that at every point in M the following equations hold

(3.4) $$h_{ce}^{\ \ x} f_b^e - h_{be}^{\ \ x} f_c^e = 2\alpha^x f_{cb},$$

where α^x is a normal bundle valued function on M.

Transvecting (3.4) with $f_y^b f_d^c$ and using (1.10) and (2.1), we have

(3.5) $$h_{ce}^{\ \ x} f_y^e = P_{zy}^{\ \ x} f_c^z + (\delta_y^x + f_y^z f_z^x) v_c,$$

where $P_{zy}^{\ \ x} = P_{zyw} g^{wx}$, $P_{zy}^{\ \ x} = h_{cb}^{\ \ x} f_z^c f_y^b$.

On the other hand, transvecting (3.4) with f_x^y and using (3.2) we have

$$\alpha^x f_x^y f_{cb} = 0,$$

which and (1.10) imply

(3.6) $$(n-A)\alpha^x f_x^y = 0.$$

Since, as already mentioned in Lemma 3.3, A is constant, we may consider only two cases;

(I) $n-A=0$, (II) $n-A\neq 0$.

(I) $n-A=0$; In such a case, since $n-A = f_{cb} f^{cb}$, we have $f_{cb}=0$, which means that M is anti-invariant submanifold of $S^{2m+1}(1)$.

(II) $n-A\neq 0$; In this case, from (3.6), we have $\alpha^x f_x^y = 0$, which and (1.10) give

(3.7) $$\|\alpha^x f_{cx}\|^2 = \|\alpha_x\|^2.$$

Differentiating (3.5) covariantly along M and using (3.1), we have

(3.8) $$(\nabla_d h_{ce}^{\ \ x}) f_y^e - h_c^{\ ex} h_{day} f_e^a = (\nabla_d P_{zy}^{\ \ x}) f_c^z - P_{zy}^{\ \ x} h_{de}^{\ \ z} f_c^e + (\delta_y^x + f_y^z f_z^x) f_{dc},$$

from which, taking the skew-symmetric part with respect to the indices c and d,

$$(3.9)\qquad -2h_{ce}^{\ x}h_{ay}^{e}f_{d}^{a}-2(\alpha_{y}h_{ce}^{\ x}+\alpha^{x}h_{cey})f_{d}^{e} = (\nabla_{d}P_{zy}^{\ x})f_{c}^{z}-(\nabla_{c}P_{zy}^{\ x})f_{d}^{z}-2(\alpha^{z}P_{zy}^{\ x}-\delta_{y}^{x}-f_{y}^{z}f_{z}^{x})f_{dc}$$

sinec the normal connection is flat.

Transvecting (3.9) with f_{w}^{c} and using (1.10), we can easily find

$$(3.10)\qquad \nabla_{d}P_{zy}^{\ x}=(f_{z}^{e}\nabla_{e}P_{wy}^{\ x})f_{d}^{w},$$

where we have used $(\nabla_{d}P_{zy}^{\ x})f_{w}^{z}=0$ which is a direct consequence of $P_{zy}^{\ x}f_{w}^{z}=0$ and (3.1). On the other hand, $P_{zy}^{\ x}=P_{yz}^{\ x}$, which and (3.10) yield

$$(3.11)\qquad (\nabla_{d}P_{zy}^{\ x})f_{c}^{z}=(\nabla_{c}P_{zy}^{\ x})f_{d}^{z},$$

$$(3.12)\qquad f_{c}^{d}\nabla_{d}P_{zy}^{\ x}=0.$$

Thus, from (3.9) and (3.11), we obtain

$$h_{ce}^{\ x}h_{ay}^{\ e}f_{b}^{a}+(\alpha_{y}h_{ce}^{\ x}+\alpha^{x}h_{cey})f_{d}^{e}=(\alpha^{z}P_{zy}^{\ x}-\delta_{y}^{x}-f_{y}^{z}f_{z}^{x})f_{dc},$$

from which, transvecting with f_{b}^{d},

$$(3.13)\qquad \begin{aligned} h_{ce}^{\ x}h_{by}^{\ e}=&(\alpha^{z}P_{zy}^{\ x}-\delta_{y}^{x}-f_{y}^{z}f_{z}^{x})(g_{cb}-v_{c}v_{b}-f_{c}^{w}f_{wb})\\ &+P_{wzy}P^{wx}_{\ \ u}f_{c}^{u}f_{b}^{z}+P_{zy}^{\ x}(v_{c}f_{b}^{z}+v_{b}f_{c}^{z})+f_{by}f_{c}^{x}\\ &+(\delta_{y}^{x}+f_{y}^{z}f_{z}^{x})v_{c}v_{b}-\alpha_{y}h_{cb}^{\ x}-\alpha^{x}h_{cby}\\ &+\alpha_{y}(v_{c}f_{b}^{x}+v_{b}f_{c}^{x})+\alpha^{x}(v_{c}f_{by}+v_{b}f_{cy})+(\alpha_{y}P_{wz}^{\ x}\\ &+\alpha^{x}P_{wzy})f_{c}^{w}f_{b}^{z}, \end{aligned}$$

where $P_{zyx}=P_{zy}^{\ w}g_{wx}$.

Transvecting g^{cb} to (3.13), we obtain

$$(3.14)\qquad \begin{aligned} h_{cb}^{\ x}h^{cb}_{\ \ y}=&(n-A)(\alpha^{z}P_{zy}^{\ x}-\delta_{y}^{x}-f_{y}^{z}f_{z}^{x})+P_{zwy}P^{zwx}\\ &+2(\delta_{y}^{x}+f_{y}^{z}f_{z}^{x})-\alpha_{y}h^{x}-\alpha^{x}h_{y}+\alpha_{y}P^{x}+\alpha^{x}P_{y}. \end{aligned}$$

where $P^{zwx}=P_{yu}^{\ x}g^{yz}g^{uw}$ and $P^{x}=P_{zy}^{\ x}g^{zy}$.

On the other hand, transvecting (3.4) with f^{cb}, we can see that

$$(3.15)\qquad h^{x}=P^{x}+(n-A)\alpha^{x},$$

which and (3.14) imply

$$h_{cbx}h^{cb}_{\ \ y}=(n-A)\alpha^{z}P_{zyx}+(2-n+A)(g_{xy}+f_{x}^{z}f_{zy})+P_{zwy}P^{zwx}-2(n-A)\alpha_{x}\alpha_{y},$$

and consequently

$$(3.16)\qquad \begin{aligned} &\|h_{cbx}\alpha^{x}\|^{2}-\|\alpha^{z}P_{zyx}\|^{2}-2\|\alpha_{x}\|^{2}\\ &=(n-A)\{\alpha^{z}\alpha^{y}\alpha^{x}P_{zyx}-\|\alpha_{x}\|^{2}-2\|\alpha_{x}\|^{4}\}. \end{aligned}$$

Differentiating (3.10) covariantly along M and using (1.13) and $(\nabla_d P_{zy}^{\ \ x})f_w^z=0$, we find

$$\nabla_d\nabla_c P_{zy}^{\ \ x}=\{\nabla_d(f_y^e\nabla_e P_{zw}^{\ \ x})\}f_c^w-(f_y^e\nabla_e P_{zw}^{\ \ x})h_{da}^{\ \ w}f_c^a,$$

from which, taking account of $K_{dcy}^{\ \ \ x}=0$ and (3,4),

$$\{\nabla_d(f_y^e\nabla_e P_{zw}^{\ \ x})\}f_c^w-\{\nabla_c(f_y^e\nabla_e P_{zw}^{\ \ x})\}f_d^w-2(\alpha^w f_y^e\nabla_e P_{zw}^{\ \ x})f_{dc}=0. \tag{3.17}$$

Transvecting (3.17) with $f_u^c f_a^z$ and using (3.12), we have

$$\{\nabla_d(f_y^e\nabla_e P_{zu}^{\ \ x})\}f_a^z=\{\nabla_a(f_y^e\nabla_e P_{zu}^{\ \ x})\}f_d^z,$$

which and (3.17) imply

$$(\alpha^w f_y^e\nabla_e P_{zw}^{\ \ x})f_{dc}=0,$$

and consequently, since $n-A\neq 0$,

$$\alpha^w f_y^e\nabla_e P_{zw}^{\ \ x}=0.$$

Transvecting this equation with f_b^y and using (1.10) and (3.10), we obtain

$$\alpha^z\nabla_d P_{zy}^{\ \ x}=0. \tag{3.18}$$

From now on we add the following assumption;

The mean curvature vector h^x is parallel, that is, $\nabla_c h^x=0$ hold on M.

Transvecting (3.8) with g^{dc} and taking account of $\nabla_c h^x=0$, (1.6) and $K_{dcy}^{\ \ \ x}=0$, we find that

$$(\nabla_d P_{zy}^{\ \ x})f^{dz}=0,$$

which and (3.10) yield

$$\nabla_c P^x=0, \tag{3.19}$$

and consequently (3.15) implies that

$$\nabla_c\alpha^x=0 \tag{3.20}$$

because of $n-A\neq 0$.

On the other hand, the Ricci tensor K_{cb} of M is given by

$$K_{cb}=ng_{cb}+h^x h_{cbx}-h_{ce}^{\ \ x}h_{bx}^{\ \ e}.$$

Hence by a simple but a little long calculation, we can easily obtain

$$K_{cb}f_y^c f_x^b=(n-A)(g_{yx}+f_y^z f_{zx}+\alpha^z P_{zyx}). \tag{3.21}$$

From this equation, we can see that the quadratic form $K_{cb}f_x^b\alpha^y\alpha^x$ is constant on M with the aid of $\alpha^x f_x^y=0$, (3.18), (3.19) and (3.20).

By the way, using (1.13) and (1.16), we can easily obtain the following identity:

$$(3.22)\qquad \|\nabla_c(\alpha_x f_b^x)\|^2=\alpha^x\alpha^y(\nabla_c f_{bx})(\nabla^c f_y^b)$$
$$=\|h_{cbx}\alpha^x\|^2-\|\alpha^z P_{xyz}\|^2-2\|\alpha_x\|^2\geqq 0$$

by virtue of (3.2) and (3.20). Thus, from (3.16) and (3.22), we have

$$(n-A)\alpha^z\alpha^y\alpha^x P_{zyx}=\|\nabla_c(\alpha_x f_b^x)\|^2+(n-A)\|\alpha_x\|^2(1+2\|\alpha_x\|^2),$$

which and (3.21) give

$$(3.23)\qquad K_{cb}f_y^c\alpha^y f_x^b\alpha^x=\|\nabla_c(\alpha_x f_b^x)\|^2+2(n-A)\|\alpha_x\|^2(1+\|\alpha_x\|^2)\geqq 0.$$

Thus we have

LEMMA 3.4. *Let M be an $(n+1)$-dimensional contact CR-submanifold with flat normal connection of $S^{2m+1}(1)$. If the equation (3.4) holds on M, and if the mean curvature vector of M is parallel, then the quadratic form $K(\alpha,\alpha)=K_{cb}f_y^c\alpha^y f_x^b\alpha^x$ is nonnegative constant on M, provided $n-A\neq 0$.*

From now on we simplify the following identity under the same assumptions stated in Lemma 3.4:

$$(3.24)\qquad \nabla^c[\{\nabla_b(f_c^x\alpha_x)\}f^{by}\alpha_y]=\{\nabla^c\nabla_b(f_c^x\alpha_x)\}f^{by}\alpha_y+\{\nabla_b(f_c^x\alpha_x)\}\{\nabla^c(f^{by}\alpha_y)\}.$$

Since the normal connection is flat, it follows that

$$\nabla_c\nabla_b f_a^x-\nabla_b\nabla_c f_a^x=-K_{cba}{}^e f_e^x,$$

which and $\nabla_c f_x^c=0$ imply

$$\nabla^c\nabla_b f_c^x=K_{ba}f^{ax}.$$

Thus (3.20) yields

$$(3.25)\qquad \{\nabla^c\nabla_b(f_c^x\alpha_x)\}f^{by}\alpha_y=K(\alpha,\alpha).$$

On the other hand, by using (1.13) with (3.2) and (3.20), we have

$$\{\nabla_b(f_c^x\alpha_x)\}\{\nabla^c(f^{by}\alpha_y)\}=(h_{bex}f_c^e\alpha^x)(h_{ay}^c f^{ba}\alpha^y),$$

which and (3.4) give

$$\{\nabla_b(f_c^x\alpha_x)\}\{\nabla^c(f^{by}\alpha_y)\}=\|\nabla_c(\alpha_x f_b^x)\|^2+2\|\alpha_x\|^2(\alpha^y h_y-\alpha^y P_y),$$

and consequently it follows from (3.15) that

$$(3.26)\qquad \{\nabla_b(f_c^x\alpha_x)\}\{\nabla^c(f^{by}\alpha_y)\}=\|\nabla_c(\alpha_x f_b^x)\|^2+2(n-A)\|\alpha_x\|^4.$$

Hence, combining (3.25) and (3.26) with (3.24), we have

$$\nabla^c[\{\nabla_b(f_c^x\alpha_x)\}f^{by}\alpha_y]=K(\alpha,\alpha)+\|\nabla_c(\alpha_x f_b^x)\|^2+2(n-A)\|\alpha_x\|^4,$$

which and Lemma 3.4 imply

LEMMA 3.5. *Let M be a compact $(n+1)$-dimensional contact CR-submanifold with flat normal connection of $S^{2m+1}(1)$. If the equations (3.4) hold on M, and if the* *mean curvature vector of M is parallel, then* $\alpha^x=0$ *on M, provided that* $n-A\neq 0$.

Finally, we calculate the Laplacian of the global function $h_{cb}^{\ x}h^{cb}_{\ \ x}$ under the same assumptions stated in Lemma 3.5:

$$\frac{1}{2}\Delta(h_{cb}^{\ x}h^{cb}_{\ \ x})=g^{de}(\nabla_d\nabla_e h_{cb}^{\ x})h^{cb}_{\ \ x}+\|\nabla_c h_{cb}^{\ x}\|^2.$$

As well known in [8], the Laplacian reduces to

$$\frac{1}{2}\Delta(h_{cb}^{\ x}h^{cb}_{\ \ x})=(n-A)(2h_{cb}^{\ x}h^{cb}_{\ \ x}+h_x h^x)+\|\nabla_c h_{ba}^{\ x}\|^2 \tag{3.27}$$

under the our assumptions. Moreover, from (3.14), it follows that $h_{cb}^{\ x}h^{cb}_{\ \ x}$ is constant on M, which and (3.27) imply that $h_{cb}^{\ x}=0$ and consequently $f_b^{\ x}=0$ with the aid of (1.10) and (3.5).

Thus we have

LEMMA 3.6. *The submanifold M stated in Lemma* 3.5 *is a* $S^{n+1}(1)$, *where* $n+1$ *is an odd number.*

Combining Theorem D, Lemma 3.1, Lemma 3.3 and Lemma 3.6, we have

THEOREM 3.7. *Let M be a compact* $(n+1)$*-dimensional contact CR-submanifold with flat normal connection of* $S^{2m+1}(1)$. *If the equations* (3.4) *hold on M, and if the mean curvature vector of M is parallel, then M is a* $S^{n+1}(1)$ *or*

$$S^1(r_1)\times\cdots\times S^1(r_{n+1}) \ \ in \ \ a \ \ S^{2n+1}(1) \ \ in \ \ S^{2m+1}(1),$$

where $\sum_{i=1}^{n+1} r_i^2=1$.

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