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SEPARATED LOCALES

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1. Introduction

Some topological properties of a topological space X can be described by the lattice theoretic properties of its open set lattice $\Omega(X)$. In this direction, the open set lattice has been recognized as a frame, i.e., a complete lattice L in which the infinite distributive law $x\wedge(\vee S)=\vee\{x\wedge s\mid s\in S\}$ holds for any $x\in L$ and any subset S of L.

Furthermore, for a continuous map $f: X\longrightarrow Y$ between topological spaces, the map $f^{-1}:\Omega(Y)\longrightarrow\Omega(X)$ preserves arbitrary joins and finite meets, and hence frame homomorphisms are defined to be those maps between frames preserving arbitrary joins and finite meets. Thus one has clearly the category **Frm** of frames and frame homomorphisms.

Moreover, the opposite category $\mathbf{Frm}^{op}$ is called the category **Loc** of locales, so frames are also in this paper called locales.

One can find references and interesting developments of the theory of locales or frames in [6].

In this paper, we introduce a concept of separated locales, namely the locales such that there is no onto frame homomorphism on the locales onto the three element chain. In fact, there have been many authors ([1], [2], [3], [4], [5], [8]) who have tried to find a suitable localic form of the T_1-separation axiom. The relationships between those are also discussed in [7].

We observe that a T_0-space X is T_1 iff the Sierpinski space S cannot be embedded in X and that $\Omega(S)$ is precisely the three element chain. This motivates the concept of separated locales, which we want to call T_1-locales; but there are already too many different kinds of T_1-locales, so we give new name separated locales.

It is shown that a T_0-space X is T_1 if $\Omega(X)$ is separated and that a sobre space X is T_1 iff $\Omega(X)$ is separated. Therefore, we have the equivalence between the category $\mathbf{Sob}_1$ of sobre T_1-spaces and the category $\mathbf{SLoc}_1$ of spatial

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separated locales. This justifies the concept of separated locales as a localic T_1-axiom. We discuss also the relationship between separated locales and other "T_1"-locales.

Finally we show that separated locales are closed under the formations of products and sublocales.

For the terminology not introduced in the paper, we refer to [6].

2. Separated locales

In the following let S denote the two point Sierpinski space $\{0, 1\}$ with the nontrivial open set $\{1\}$ and 3 denote the three element chain $\{0, \frac{1}{2}, 1\}$, which is clearly the open set lattice $\Omega(S)$.

If a T_0-space X is not a T_1-space, then there exist $a, b \in X$ such that every open neighborhood of a contains b, but there is an open neighborhood N of b with $a \notin N$, and hence the subspace $\{a, b\}$ of X is homeomorphic with S. Since S is T_0 but not T_1, a T_0-space is a T_1-space iff the Sierpinski space S can not be embedded in the space.

Using the above observation, we introduce another concept of "T_1-locales".

DEFINITION 2.1. A locale A is said to be *separated* if 3 is not a sublocale of A, equivalently there is no onto frame homomorphism $A \longrightarrow 3$.

Using the open set lattice functor $\Omega : \mathbf{Top} \longrightarrow \mathbf{Loc}$, one has immediately the following:

PROPOSITION 2.2. *A T_0-space X is a T_1-space if $\Omega(X)$ is separated.*

In the following, the functor given by $C(\_, S) : \mathbf{Top} \longrightarrow \mathbf{Loc}$ will be simply denoted by C. It is well known that $\Omega, C : \mathbf{Top} \longrightarrow \mathbf{Loc}$ are naturally isomorphic via $(\eta_X : C(X) \longrightarrow \Omega(X))_{X \in \mathbf{Top}}$, where $\eta_X(f) = f^{-1}(1)$.

It is well known [6] that the functor $\Omega \equiv C : \mathbf{Top} \longrightarrow \mathbf{Loc}$ induces an equivalence between the category **Sob** of sobre spaces and continuous maps and the full subcategory **SLoc** of **Loc** determined by spatial locales.

We recall that a map $f : X \longrightarrow Y$ between topological spaces is an embedding iff it is 1−1 initial, and that a topological space X is a T_0-space iff the source $(f : X \longrightarrow S)$ of all continuous maps on X to S is an initial mono-source in the category **Top** of topological spaces and continuous maps.

Using the above, one has,

LEMMA 2.3. *Let $f:X\longrightarrow Y$ be a continuous map. If X is a T_0-space and $C(f):C(Y)\longrightarrow C(X)$ is onto, then f is an embedding.*

The following theorem justifies the concept of separated locales as that of T_1-locales.

THEOREM 2.4. *A sobre space X is a T_1-space iff $\Omega(X)$ is separated.*

PROOF. Because of Proposition 2.2, it remains to show that for a sobre T_1-space X, $\Omega(X)$ is separated. Suppose $\Omega(X)$ is not separated, then there is an onto frame homomorphism $h:\Omega(X)\longrightarrow 3$, and hence we may regard h as an onto homomorphism on $C(X)$ onto $C(S)$. Since $C:$ **Sob**$\longrightarrow$**SLoc** is full, there is a continuous map $f:S\longrightarrow X$ with $C(f)=h$. Thus by the above lemma, $f:S\longrightarrow X$ is an embedding, so that X is not a T_1-space. This completes the proof.

REMARK 2.5. (1) For a T_0-space $X, \Omega(X)$ is separated iff the sobrification πX of X is a T_1-space, because $\Omega(X)$ and $\Omega(\pi X)$ are isomorphic.

(2) Let X be an infinite space with the cofinite topology, then the sobrification πX is given as follows: $\pi X=X\cup\{\omega\}$ $(\omega\notin X)$, and $\{\pi X-F\mid F:$ finite subset of $X\}\cup\{\phi\}$ is the topology of πX. Thus πX is not T_1 and hence $\Omega(X)$ is not separated.

More directly one can see $\Omega(X)$ is not separated as follows: pick $x_0\in X$ and define $g:\Omega(X)\longrightarrow 3$ by $g(U)=1$ iff $x_0\in U$; $g(U)=\frac{1}{2}$ iff $x_0\notin U$, $U\neq\phi$; $g(U)=0$ iff $U=\phi$. Then it is easy to show that g is an onto frame homomorphism.

NOTATION. The full subcategory of **Top** determined by sobre T_1-spaces is denoted by **Sob**$_1$ and **SLoc**$_1$ denotes the full subcategory of **Loc** determined by spatial separated locales.

Using the above theorem, one has $C($**Sob**$_1)=$**SLoc**$_1$ and hence the equivalence $C:$ **Sob**$\longrightarrow$**SLoc** induces the equivalence between **Sob**$_1$ and **SLoc**$_1$. Thus one has,

THEOREM 2.6. *The categories* **Sob**$_1$ *and* **SLoc**$_1$ *are equivalent.*

The following is due to Isbell [5] and Fourman [3].

DEFINITION 2.7. A locale L is said to be *unordered* if for any frame homomorphisms $f, g:L\longrightarrow A$, $f\leqslant g$ implies $f=g$, where $f\leqslant g$ means $f(x)\leqslant g(x)$ for all $x\in L$.

REMARK. Unordered locales are called T_1 locales by Fourman [3] and T_u locales by Johnstone [6].

It is known that a topological space X is T_1 if $\Omega(X)$ is unordered and that every regular locale is unordered.

PROPOSITION 2.8. *Every unordered locale is separated.*

PROOF. Suppose a locale A is not separated, then there is an onto frame homomorphism $f: A \longrightarrow 3$. Define $g: 3 \longrightarrow 3$ by $g(0)=0$, $g(\frac{1}{2})=g(1)=1$. Then g is a frame homomorphism and $f \leqslant g \circ f$ but $f \neq g \circ f$. Thus A is not unordered. This completes the proof.

REMARK 2.9. (1) Every regular locale is separated.

(2) There is a Hausdorff space X such that $\Omega(X)$ is not unordered (see[6]). Since every Hausdorff space is sobre, $\Omega(X)$ is by Theorem 2.4 separated.

(3) We recall that a locale A is subfit [4] or conjunctive [10], if for each two elements a, b of A with $a \not\leq b$, there is an element $c \in A$ such that $a \vee c = 1, b \vee c \neq 1$. It is known [10] that for a T_1-space $X, \Omega(X)$ is always subfit but the converse need not be true. In [7], it is shown that there is an unordered locale and hence a separated locale which is not subfit.

In the remainder of the section we consider some permanence properties of separated locales. Indeed, separated locales are hereditary and productive.

THEOREM 2.10. *Suppose $(f_i: A_i \longrightarrow A)_{i \in I}$ is a sink of frame homomorphisms such that $\bigcup_{i \in I} f_i(A_i)$ generates A. If each A_i is separated, then A is again separated.*

PROOF. Take any frame homomorphism $f: A \longrightarrow 3$. Since each A_i is separated, $f \circ f_i : A_i \longrightarrow A \longrightarrow 3$ is not onto. Thus for any $x \in A_i$, $f \circ f_i(x) \neq \frac{1}{2}$. For any $a \in A$, there is a family $(J_\lambda)_{\lambda \in \Lambda}$ of finite subsets of I and for any $\lambda \in \Lambda$ and any $j \in J_\lambda$, there is $x_{\lambda j} \in A_j$ such that $a = \bigvee_{\lambda \in \Lambda} (\bigwedge_{j \in J_\lambda} f_j(x_{\lambda j}))$, for $\bigcup_{i \in I} f_i(A_i)$ generates the frame A. Therefore,

$$\begin{aligned} f(a) &= f(\bigvee_{\lambda \in \Lambda} (\bigwedge_{j \in J_\lambda} f_j(x_{\lambda j})) \\ &= \bigvee_{\lambda \in \Lambda} (\bigwedge_{j \in J_\lambda} f(f_j(x_{\lambda j}))) \neq \frac{1}{2}. \end{aligned}$$

Hence f is not onto.

COROLLARY 2.11. *A sublocale of a separated locale is again separated and a product of separated locales is separated.*

PROOF. The first part is immediate from the fact that sublocales of a locale are given by frame homomorphic images. For the second half, let $(A_i)_{i\in I}$ be a family of separated locales and $(\prod_{i\in I} A_i, (p_i)_{i\in I})$ the product of $(A_i)_{i\in I}$ in the category **Loc**. Since $((p_i)_{i\in I}, \prod_{i\in I} A_i)$ is the coproduct of (A_i) in the category **Frm** of frames, $\bigcup_{i\in I} p_i(A_i)$ generates $\prod_{i\in I} A_i$. Thus one has the result.

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