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CHARACTERIZATIONS OF SQ-CLOSED SPACES

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1. Introduction

A topological space is said to be *SQ*-closed if its continuous image in any Hausdorff space is sequentially closed [4]. Characterizations of *SQ*-closed spaces are obtained together with known characterizations presented in [3]. All topological spaces in this paper are assumed to be Hausdorff. Following the notation of [2], let S denote a class of first countable topological spaces containing the class of first countable Hausdorff completely normal and fully normal spaces. The one-point compactification of the positive integers N with discrete topology will be denoted by $\bar{N}$. The notation $T(n)$ will be used to denote the set $\{k \mid k \geq n,\ n \in N\}$.

2. Preliminaries

The following definition and theorems are given for future reference.

DEFINITION 2.1. The sequence $\{x_n\}$ r-converges (accumulates) to a point $x \in X$ if for each $V \in N(x)$, $\{x_n\}$ is eventually (frequently) in $\bar{V}$. [1]

DEFINITION 2.2. A function $f: X \to Y$ is weakly continuous at each point $x \in X$ if for every open $V \in N(f(x))$, there exists a $U \in N(x)$ such that $f(U) \subset \bar{V}$. [3]

DEFINITION 2.3. The graph of a function $f: X \to Y$, denoted $G(f)$, is strongly closed if for $f(x) \neq y$, there exist open sets U and V containing x and y, respectively, such that $f(U) \cap \bar{V} = \phi$. [1]

THEOREM 2.4. *A topological space X is SQ-closed if and only if every sequence in X has an r-accumulation point.* [4, Th. 4.1]

THEOREM 2.5. *For a function $f: X \to Y$, the following are equivalent:*

1) *f is weakly continuous;*

2) *if $\{x_\alpha\} \to x$, then $\{f(x_\alpha)\}$ r-converges to $f(x)$*; [1, Th. 6]

3) *for every open set V in Y, $f^{-1}(V) \subset [f^{-1}(\bar{V})]^0$.* [3]

3. Main Results

THEOREM 3.1. *If Y is SQ-closed and $f: X \to Y$ has a strongly closed graph, then $\{x_\alpha\} \to x$ implies the net $\{f(x_\alpha)\}$ r-accumulates to $f(x)$.*

PROOF. Let $\{x_\alpha\} \to x$. Then since Y is SQ-closed we have that $\{f(x_\alpha)\}$ r-accumulates to some point $y \in Y$. Suppose that $f(x) \neq y$. Then, since $G(f)$ is strongly closed, there exist open sets U and V containing x and y, respectively, such that $f(U) \cap \overline{V} = \phi$. Because $\{x_\alpha\}$ is eventually in U, we have that $\{f(x_\alpha)\}$ is eventually in $f(U)$. But $\{f(x_\alpha)\}$ having y as an r-accumulation point implies that $\{f(x_\alpha)\}$ is frequently in $\overline{V}$, contradicting the fact that $f(U) \cap \overline{V} = \phi$. Therefore, we must have $f(x) = y$.

THEOREM 3.2. *Let $f: X \to Y$ be any function where Y is SQ-closed and X is first countable. If $G(f)$ is strongly closed, then f is weakly continuous.*

PROOF. Suppose there exists an open V in Y such that $f^{-1}(V) \not\subset [f^{-1}(\overline{V})]^0$. This implies there exists an $x \in f^{-1}(V)$ such that $x \in X - [f^{-1}(\overline{V})]^0 = \overline{X - f^{-1}(\overline{V})}$. There now exists a sequence $\{x_n\}$ in $X - f^{-1}(\overline{V})$ such that $x_n \to x$. By Theorem 3.1, $\{f(x_n)\}$ must have non-empty intersection with $\overline{V}$ since $V \in N(f(x))$. This contradiction establishes the fact that f is weakly continuous.

THEOREM 3.3. *If for every topological space $X \in S$ each mapping of X into Y with strongly closed graph is weakly continuous, then Y is SQ-closed.*

PROOF. Suppose Y is not SQ-closed. Then there exists a sequence $\{y_n\}$ in Y that has no r-accumulation point. Let $X = N \cup \{\infty\} = \overline{N}$. The space $\overline{N}$ is fully normal and completely normal and thus belongs to S. Selecting $b \in Y$ we define a function $f: X \to Y$ by $f(n) = y_n$, and $f(\infty) = b$. We now show that $G(f)$ is strongly closed.

Case 1 : Suppose $f(n) \neq y$. Then since Y is Hausdorff, there exists $V \in N(y)$ such that $f(n) \not\in \overline{V}$. Therefore, $f(\{n\}) \times \overline{V} = \phi$.

Case 2 : Suppose $f(\infty) \neq y$. Then since $\{y_n\}$ does not r-accumulate to y, there exists $V \in N(y)$, $b \not\in \overline{V}$, such that $T\{y_n\} = f(T(n))$ is eventually outside of $\overline{V}$. Since $[T(n) \cup \{\infty\}] \in N(\infty)$, we have $f[T(n) \cup \{\infty\}] \cap \overline{V} = \phi$ for sufficiently large n. In any case, we have that $G(f)$ is strongly closed.

Additionally, we can see that f is not weakly continuous. In particular, consider the sequence $\{x_n\}$, where $x_n = n$. Obviously, we have $x_n \to \infty$, but since

$\{f(x_n)\} = \{f(n)\} = \{y_n\}$ has no r-accumulation point, it cannot r-converge. Appealing to Theorem 2.5 (2), the theorem follows from contraposition.

COROLLARY 3.4. *A Hausdorff space Y is SQ-closed if and only if for every topological space X which belongs to S, each mapping of X into Y with strongly closed graph is weakly continuous.*

COROLLARY 3.5. *For a topological Hausdorff space Y, the following are equivalent:*

1) *Y is SQ-closed;*

2) *For every first countable topological space X, each mapping of X into Y with strongly closed graph is weakly continuous;*

3) *Each mapping of $\overline{N}$ into Y with strongly closed graph is weakly continuous.*

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