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PRODUCTS OF SEVERAL GENERALIZED LAGUERRE POLYNOMIALS

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1. Introduction

Let

$$(1.1)\qquad L_n^{(\alpha)}(x)=\sum_{r=0}^{n}\frac{(-1)^r(\alpha+1)_n}{(n-r)!\,(1+\alpha)_r}\,\frac{x^r}{r!}$$

denote the Laguerre polynomials of degree n. The expansion formula

$$(1.2)\qquad L_{n_1}^{(\alpha_1)}(a_1x)\cdots L_{n_p}^{(\alpha_p)}(a_px)=\sum_{s=0}^{n_1+\cdots+n_p}B_s^{(n_1,\cdots,n_p)}\,L_s^{(\beta)}(x)$$

where in the coefficients $B_s^{(n_1,\cdots,n_p)}$ in terms of Lauricella's hypergeometric function F_A of $p+1$ variables, are given by

$$(1.3)\qquad B_s^{(n_1,\cdots,n_p)}=\frac{(\alpha_1+1)_{n_1}}{n_1!}\cdots\frac{(\alpha_p+1)_{n_p}}{n_p!}$$

$\times F_A(\beta+1,\ -n_1,\ \cdots,\ -n_p,\ -s:\alpha_1+1,\ \cdots,\ \alpha_p+1,\ \cdots,\ \beta+1: a_1,\ \cdots,\ a_p,\ 1)$ is due to Erdelyi [2]. In the year 1962 Carlitz [1] obtained a generating function for the coefficients $B_s^{(n_1,\cdots,n_p)}$ in the form

$$(1.4)\qquad \sum_{n_1=0}^{\infty}\cdots\sum_{n_p=0}^{\infty}B_s^{(n_1,\cdots,n_p)}\,u_1^{n_1}\cdots u_p^{n_p}$$

$$=\frac{(1-u_1)^{-\alpha_1-1}\cdots(1-u_p)^{-\alpha_p-1}}{\left(1+\dfrac{a_1u_1}{1-u_1}+\cdots+\dfrac{a_pu_p}{1-u_p}\right)^{\beta+s+1}}\left(\frac{a_1u_1}{1-u_1}+\cdots+\frac{a_pu_p}{1-u_p}\right)^s.$$

It may be of interest to obtain an analogous expansion for the product $\Gamma_{n_1}^{c_1}(a_1x,y_1,r_1)\cdots\Gamma_{n_p}^{c_p}(a_px,y_p,r_p)$, where $\Gamma_n^c(x,y,r)$ denotes the generalized Laguerre polynomial given by

$$(1.5)\qquad \Gamma_n^c(x,y,r)=\sum_{k=0}^{n}\frac{(-y)^{n-k}(c+rk)_{n-k}\,x^k}{(n-k)!\,k!}$$

in which the parameters c, y and r are unrestricted in general. The polynomials $\Gamma_n^c(x,y,r)$ as given by (1.5) are included in the definition of more general classes of polynomials $\{g_n^c(x,r,s)\}$ and $\{f_n^c(x,y,r,m)\}$ introduced recently by R. Panda ([3] : see also [7]) and the authors [5] respectively.

In this paper we first obtain the expansion formula

$$\Gamma_{n_1}^{c_1}(a_1 x, y_1, r_1)\cdots\Gamma_{n_p}^{c_p}(a_p x, y_p, r_p) = \sum_{s=0}^{n_1+\cdots+n_p} C_s^{(n_1,\cdots,n_p)}\,\Gamma_s^c(x,y,r) \tag{1.6}$$

where the coefficients $C_s^{(n_1,\cdots,n_p)}$ are given by equation (2.4). We then derive the generating relation

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_p=0}^{\infty} C_s^{(n_1,\cdots,n_p)}\, u_1^{n_1}\cdots u_p^{n_p} = (1+y_1u_1)^{-c_1}\cdots(1+y_pu_p)^{-c_p}\, U^s\, V^{c+rs}, \tag{1.7}$$

where U and V are defined by

$$\begin{cases} U = \dfrac{a_1u_1}{(1+y_1u_1)^{r_1}} + \cdots + \dfrac{a_pu_p}{(1+y_pu_p)^{r_p}} \\ V = 1 + yUV^r \end{cases} \tag{1.8}$$

In view of the relation

$$L_n^{(\alpha)}(x) = \Gamma_n^{\alpha+1}(-x, -1, 1) \tag{1.9}$$

the expansion formula (1.6) and the accompanying generating relation (1.7) obviously provide extensions of (1.2) and (1.4) to which they would reduce when the various parameters involved therein are particularized in accordance with (1.9).

A further extension of (1.6) and (1.7) is also given in the last section of this paper.

2. Proof of (1.6) and (1.7)

By making use of the pair of inverse series relation given by us [6] it is easy to deduce that

$$\frac{x^n}{n!} = \sum_{k=0}^{n} \frac{(-y)^{n-k}(-c-rk)(1-c-rn)_{n-k-1}}{(n-k)!}\,\Gamma_k^c(x,y,r) \tag{2.1}$$

From (1.5) and (2.1) it follows that

$$\begin{aligned}(2.2)\quad & \Gamma_{n_1}^{c_1}(a_1x,y_1,r_1)\cdots\Gamma_{n_p}^{c_p}(a_px,y_p,r_p) \\ &= \sum_{k_1=0}^{n_1}\cdots\sum_{k_p=0}^{n_p}\left[\prod_{j=1}^{p}\frac{(c_j+r_jk_j)_{n_j-k_j}}{k_j!(n_j-k_j)!}\,a_j^{k_j}(-y_j)^{n_j-k_j}\right] \\ &\quad\times\sum_{s=0}^{K}\frac{K!(-y)^{K-s}(-c-rs)(1-c-rK)_{K-s-1}}{(K-s)!}\,\Gamma_s^c(x,y,r)\end{aligned}$$

where for the sake of convenience, $k_1+k_2+\cdots+k_p$ has been abbreviated as K.

A rearrangement of the series on the right hand side of (2.2) would transform it to

$$\Gamma_{n_1}^{c_1}(a_1x, y_1, r_1)\cdots\Gamma_{n_p}^{c_p}(a_px, y_p, r_p) \sum_{s=0}^{n_1+\cdots+n_p} C_s^{(n_1,\cdots,n_p)}\, \Gamma_s^{c}(x, y, r) \tag{2.3}$$

where

$$C_s^{(n_1,\cdots,n_p)} = \sum_{k_1=0}^{n_1}\cdots\sum_{k_p=0}^{n_p}\left[\prod_{j=1}^{p}\frac{(c_j+r_jk_j)_{n_j-k_j}\,a_j^{k_j}(-y_j)^{n_j-k_j}}{k_j!\,(n_j-k_j)!}\right] \times\frac{K!\,(-y)^{K-s}(-c-rs)(1-c-rK)_{K-s-1}}{(K-s)!} \tag{2.4}$$

In order to obtain the generating function for the coefficients $C_s^{(n_1,\cdots,n_p)}$, we observe that

$$\begin{aligned}
&\sum_{n_1=0}^{\infty}\cdots\sum_{n_p=0}^{\infty} C_s^{(n_1,\cdots,n_p)}\, u_1^{n_1}\cdots u_p^{n_p} = (1+y_1u_1)^{-c_1}\cdots(1+y_pu_p)^{-c_p}\\
&\times\sum_{k_1=0}^{\infty}\cdots\sum_{k_p=0}^{\infty}\frac{K!\,(-y)^{K-s}(-c-rs)(1-c-rK)_{K-s-1}}{(K-s)!\,k_1!\cdots k_p!}\\
&\qquad\times\frac{(a_1u_1)^{k_1}}{(1+y_1u_1)^{r_1k_1}}\cdots\frac{(a_pu_p)^{k_p}}{(1+y_pu_p)^{r_pk_p}}\\
&=(1+y_1u_1)^{-c_1}\cdots(1+y_pu_p)^{-c_p}\\
&\quad\times\sum_{k=s}^{\infty}\frac{(-y)^{k-s}(-c-rs)(1-c-rk)_{k-s-1}}{(k-s)!}\,U^k,\\
&=(1+y_1u_1)^{-c_1}\cdots(1+y_pu_p)^{-c_p}\,U^s\\
&\quad\times\sum_{k=0}^{\infty}\frac{c+rs}{c+rs+rk}\binom{c+rs+rk}{k}(yU)^k,
\end{aligned}$$

where U is given by (1.8). Now if we make use of the well known expansion (cf. [4 : p. 348, problem 212])

$$x^a = \sum_{k=0}^{\infty}\frac{a}{a+bk}\binom{a+bk}{k}z^k,\quad z=(x-1)x^{-b} \tag{2.5}$$

we are readily led to the desired generating function (1.7).

3. Extension of (1.6) and (1.7)

In this section we consider the following particular case of the class of polynomials $\{f_n^c(x, y, r, m)\}$ introduced in [5] :

$$\Gamma_n^c(x, y, r, m) = \sum_{k=0}^{[n/m]}(-y)^k\frac{(c+rn-rmk)_k}{(n-mk)!\,k!}\,x^{n-mk}, \tag{3.1}$$

and its inverse relation

$$(3.2)\qquad \frac{x^n}{n!}=\sum_{k=0}^{[n/m]}(-y)^k\,\frac{(1-c-rn)_{k-1}(rmk-c-rn)}{k!}\,\Gamma^c_{n-mk}(x,y,r,m)$$

which is an easy consequence of the corresponding relation for $f^c_n(x,y,r,m)$ derived in [6].

For the sake of brevity we shall use the following notations:

$$(3.3)\qquad \begin{cases} n_1+n_2+\cdots+n_p=N \\ k_1+k_2+\cdots+k_p=K \\ \left[\dfrac{n_i}{m}\right]=n^*_i,\ i=1,2,\cdots,p\,;\ \left[\dfrac{n}{m}\right]=n^*,\ \left[\dfrac{N}{m}\right]=N^* \\ W=\left[\dfrac{a_1u_1}{(1+y_1u_1^m)^{r_1}}+\cdots+\dfrac{a_pu_p}{(1+y_pu_p^m)^{r_p}}\right] \end{cases}$$

We now observe that (3.1) and (3.2), when combined, leads us to

$$\Gamma^{c_1}_{n_1}(a_1x,y_1,r_1,m)\cdots\Gamma^{c_p}_{n_p}(a_px,y_p,r_p,m)$$
$$=\sum_{k_1=0}^{n^*_1}\cdots\sum_{k_p=0}^{n^*_p}\left[\prod_{j=1}^{p}\frac{(c_j+r_jn_j-r_jmk_j)k_j\,a_j^{n_j-mk_j}(-y_j)^{k_j}}{k_j!\,(n_j-mk_j)!}\right]$$
$$\times\sum_{s=0}^{N^*-K}\frac{(N-mK)!\,(-y)^s(1-c-rN+rmK)_{s-1}(rms-c-rN+rmK)}{s!}$$
$$\times\Gamma^c_{N-mk-ms}(x,y,r,m)$$

which in view of the easily derivable relation

$$(3.4)\qquad \sum_{k=0}^{r}\sum_{s=0}^{r-k}\frac{A(r,s,k)}{s!}=\sum_{s=0}^{r}\sum_{k=0}^{r}\frac{A(r,r-k-s,k)}{(r-k-s)!}$$

gives us

$$\Gamma^{c_1}_{n_1}(a_1x,y_1,r_1,m)\cdots\Gamma^{c_p}_{n_p}(a_px,y_p,r_p,m)$$
$$(3.5)\qquad =\sum_{s=0}^{N^*}D_s^{(n_1,\cdots,n_p)}\,\Gamma^c_{N+ms-mN^*}(x,y,r,m)$$

where the coefficients $D_s^{(n_1,\cdots,n_p)}$ are given by

$$D_s^{(n_1,\cdots,n_p)}=\sum_{k_1=0}^{n^*_1}\cdots\sum_{k_p=0}^{n^*_p}\left[\prod_{j=1}^{p}\frac{(c_j+r_jn_j-r_jmk_j)_{k_j}a_j^{n_j-mk_j}(-y_j)^{k_j}}{k_j!\,(n_j-mk_j)!}\right]$$
$$\times(-y)^{N^*-K-s}\frac{(N-mK)!\,(1-c-rN+rmK)_{N^*-K-s-1}(-c-rms-rN+rmN^*)}{(N^*-K-s)!}$$

Hence

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_p=0}^{\infty}D_s^{(n_1,\cdots,n_p)}\,u_1^{n_1}\cdots u_p^{n_p}=(1+y_1u_1^m)^{-c_1}\cdots(1+y_pu_p^m)^{-c_p}$$

$$\times \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} \frac{N!\, y^{N^*-s}(c+rms+rN-rmN^*)(1-c-rN)_{N^*-s-1}}{n_1!\cdots n_p!(N^*-s)!}$$

$$\times(-1)^{N^*-s-1}\left(\frac{a_1u_1}{(1+y_1u_1^m)^{r_1}}\right)^{n_1}\cdots\left(\frac{a_pu_p}{(1+y_pu_p^m)^{r_p}}\right)^{n_p}$$

$$=(1+y_1u_1^m)^{-c_1}\cdots(1+y_pu_p^m)^{-c_p}$$

$$\times\sum_{n=ms}^{\infty}\frac{(1-c-rn)_{n^*-s-1}(c+rms+rn-rmn^*)}{(n^*-s)!}\,y^{n^*-s}(-1)^{n^*-s-1}\,W^n.$$

Now since

$$\text{(3.7)} \qquad \sum_{n=ms}^{\infty} A(n,n^*,m,s)=\sum_{n=s}^{\infty}\sum_{j=0}^{m-1}A(mn+j,n,m,s)$$
$$=\sum_{j=0}^{m-1}\sum_{n=0}^{\infty}A(mn+ms+j,n+s,m,s),$$

the right hand side of the above expression may be put in the form

$$(1+y_1u_1^m)^{-c_1}\cdots(1+y_pu_p^m)^{-c_p}\,W^{ms}$$
$$\times\sum_{j=0}^{m-1}W^j\sum_{n=0}^{\infty}\frac{c+rms+rj}{c+rms+rj+rmn}\binom{c+rms+rj+rmn}{n}(yW^m)^n$$

The inner series of the last expression can be summed up with the help of (2.5), as a result of which, we finally get the generating function

$$\text{(3.8)} \qquad \sum_{n_1=0}^{\infty}\cdots\sum_{n_p=0}^{\infty}D_s^{(n_1,\cdots,n_p)}\,u_1^{n_1}\cdots u_p^{n_p}$$
$$=(1+y_1u_1^m)^{-c_1}\cdots(1+y_pu_p^m)^{-c_p}\,W^{ms}\,\eta^{c+rms}\,\frac{1-W^m\eta^{rm}}{1-W\eta^r}$$

where W is as given in (3.3) and

$$\text{(3.9)} \qquad \eta=1+y\eta^{rm}\,W^m.$$

When $m=1$ the expansion (3.5) and the generating relation (3.8) would evidently reduce to the expansion (1.6) and the corresponding generating relation (1.7).

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REFERENCES

[1] L. Carlitz, *The product of several Hermite or Laguerre polynomials*, Montash. Math. 66 (1962), 393~396.

[2] A. Erdelyi, *On some expansions in Laguerre polynomials*, J. London Math. Soc. 13(1938), 154~156.

[3] R. Panda, *On a new class of polynomials*, Glasgow Math. J. 18(1977), 105~108.

[4] G. Polya and G. Szego, *Problems and Theorems in Analysis*, Vol. I (Translated from the German by D. Aeppli), Springer-Verlag, New York, Heidelberg and Berlin, 1972.

[5] J.P. Singhal and Savita Kumari, *On a unification of generalized Humbert and Laguerre polynomials*, Jnanabha (Prof. Erdelyi Memorial Volume) 9/10 (1980), To appear.

[6] J.P. Singhal and Savita Kumari, *On an extension of an inverse series relation*, Bull. Inst. Math. Acad. Sinica 10(1982), 171~175.

[7] R. Srivastava, *New generating functions for a class of polynomials*, Boll. Un. Math. Ital. (5) 17A(1980), 1~4.