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A RIESZ REPRESENTATION THEOREM IN 2-SEMI-INNER PRODUCT SPACES

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1. Introduction

With the aim of carrying over Hilbert space type arguments to the theory of Banach spaces, Lumer [7] introduced the concept of semi-inner product space with a more general axiom system than that of Hilbert space. Giles [6] imposed some restrictions on semi-inner product spaces and obtained the Riesz representation theorem for semi-inner product spaces. Motivated by the concepts of semi-inner product space due to Lumer [7] and 2-inner product space introduced in [1], Siddiqui and Rizvi [9] introduced the concept of 2-semi-inner product space. In an earlier paper [8], we studied the concepts of orthogonality and Gateax differentiability in 2-semi-inner product spaces and gave a characterization of strict convexity in terms of 2-semi-inner product. In the present paper we prove an analogue of the Riesz representation theorem in 2-semi-inner product space similar to the one proved by Giles [6] in semi-inner product spaces.

2. Preliminaries

The following concept of 2-normed space was introduced in [4] :

Let E be a vector space with $\dim(E)>1$ and $\|\cdot,\cdot\|$ a real function on $E\times E$ which satisfies the following axioms:

(a) $\|x,y\|=0$ iff x and y are linearly independent;

(b) $\|x,y\|=\|y,x\|$;

(c) $\|\lambda x,y\|=|\lambda|\ \|x,y\|,\ \lambda\in \boldsymbol{R}$;

(d) $\|x+y,z\|\leq\|x,z\|+\|y,z\|$.

We call $\|\cdot,\cdot\|$ a 2-norm on E, and E, equipped with $\|\cdot,\cdot\|$, is called a 2-normed space written as $(E,\ \|\cdot,\cdot\|)$.

For non-zero vectors $x,y\in E$, we denote by $V(x,y)$ the subspace of E generated by x and y. A 2-normed space $(E,\ \|\cdot,\cdot\|)$ is said to be strictly convex if

$$\|x+y,z\|=\|x,z\|+\|y,z\|,\ \ \|x,z\|=1,\ \ \|y,z\|=1$$

and $z \notin V(x,y) \Longrightarrow x=y$ ([2]).

We recall the following definition from [9] :

Let E be a vector space with $\dim(E)>1$ and $[\cdot,\cdot]$ a real function on $E\times E \times E$ which satisfies the following conditions:

(S_1) $[x+x_1,\ y/z]=[x,\ y/z]+[x_1,\ y/z],\ x,x_1,y,z\in E$;

(S_2) $[\lambda x,\ y/z]=\lambda[x,\ y/z],\ \lambda\in \boldsymbol{R},\ x,y,z\in E$;

(S_3) $[x,\ y/z]>0$ if x and y are linearly independent.

(S_4) $|[x,\ y/z]|\leq[x,\ x/z]^{1/2}\ [y,\ y/z]^{1/2}$.

$[\cdot,\cdot/\cdot]$ is called a 2-semi-inner on E. E, equipped with $[\cdot,\cdot/\cdot]$, is called a 2-semi-inner product space (henceforth abbreviated to 2-s. i. p. space).

Every 2-normed space can be made into a 2-s. i. p. space and a 2-s. i. p. space is a 2-normed space with

$$\|x,z\|=[x,\ x/z]^{1/2}$$

provided $[x,\ x/z]=[z,\ z/x]$ ([9]).

A sequence $\{x_n\}$ in a 2-normed space E is called a Cauchy sequence if there exist $y,\ z\in E$ such that y and z are linearly independent with

$$\lim_{n\to\infty}\|x_n-x_m,\ y\|=\lim_{n\to\infty}\|x_n-x_m,\ z\|=0$$

A sequence $\{x_n\}$ in E is called a convergent sequence if there is an $x\in E$ such that

$$\lim_{n\to\infty}\|x_n-x,\ y\|=0$$

for all $y\in E$. We then write $x_n\to x$ as $n\to\infty$.

A 2-normed space E is said to be complete if every Cauchy sequence converges in E.

Let $c\in E$ and $[c]$ denote the vector space generated by c.

Let E be a 2-normed space. A map $f: E\times\{[c]\}\to \boldsymbol{R}$ is a linear 2-functional on $E\times\{[c]\}$ if for $a,b\in E$ the following holds:

(i) $f(a+b,c)=f(a,c)+f(b,c)$

(ii) $f(\lambda a,b)=\lambda f(a,b)$, where λ belongs to $\boldsymbol{R}$.

Remark: Let f be a linear 2-functional defined on $E\times\{[z]\}$ where $[z]$ is the vector subspace generated by $z\in E$. Then f is continuous at (a,z) if given $\varepsilon>0$, there is a $\delta>0$ such that $|f(a,z)-f(b,z)|<\varepsilon$ whenever $\|a-b,\ z\|<\delta$.

As in [8], we consider here 2-s. i. p. spaces with the homogeneity property: $[x,\ \lambda y/z]=\lambda[x,\ y/z],\ \lambda\in\boldsymbol{R}$.

3. Main results

We start by extending the definition of uniform convexity to 2-normed spaces as follows:

DEFINITION 3.1. A 2-normed space is uniformly convex if given $\varepsilon>0$ there exists $\eta>1$ such that $x,y\in E$, $\|x,z\|=\|y,z\|=1$ for all $z\in E-V(x,y)$, $\|x-y,\ z\|>2\varepsilon \Longrightarrow \|x+y,\ z\|\leq 2\eta$.

REMARK 3.2. The above definition is equivalent to the following:

Given $\varepsilon>\varepsilon'>0$, there exists $\delta>0$ such that if $x,y\in E$ with $\|x,z\|\leq\frac{1}{\varepsilon'}$, $\|y,z\|\leq\frac{1}{\varepsilon'}$ for all $z\in E-V(x,y)$ and $|\|x,z\|-\|y,z\||\leq\varepsilon'$ and $\|x-y,\ z\|\geq\varepsilon$, then

$$\|x+y,\ z\|\leq\|x,z\|+\|y,z\|-\delta.$$

It can be shown that uniform convexity $\Longrightarrow$ strict convexity.

LEMMA 3.3. *Let N be a proper closed vector subspace in a 2-s.i.p. space E which is uniformly convex, and let $y\notin N$. Then there exists a unique nonzero vector $x_0\in N$ such that*

$$\|y-x_0,\ z\|=\inf\{\|y-x,\ z\|\ ;\ x\in N\} \qquad (*)$$

for all $z\in E-V(y,N)$.

PROOF. Let $0<r=\inf\{\|y-x,\ z\| : x\in N\}$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in E such that

$$\|y-x_n,\ z\|=r_n\downarrow r \text{ (increasing to } r).$$

We assert that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Suppose not, and let ρ be a positive real number such that $r_n\leq\rho$ for all n.

Let $\varepsilon>0$ be given and let $\varepsilon'<\min\left\{\frac{1}{\rho},\ \varepsilon\right\}$. Then there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that for some positive integer J_1 we have

$$\|x_{n_j}-x_{n_{j+1}},\ z\|>\varepsilon \text{ for all } j>J_1$$

for all $z\in E$ except possibly for one element $z_0\in E$.

Hence,

(1) $\|(y-x_{n_j})-(y-x_{n_{j+1}}),\ z\|>\varepsilon$ for all $j>J_1$.

Next, choose $J_2\geq J_1$ such that

(2) $|\|y-x_{n_j},\ z\|-\|y-x_{n_{j+1}},\ z\||=|r_{n_j}-r_{n_{j+1}}|\leq\varepsilon'$

for all $j\geq j_2$. We also have

(3) $\|y-x_{n_j},\ z\|,\ \|y-x_{n_{j+1}},\ z\|\leq\rho<\frac{1}{\varepsilon}$, for all j.

Combining (1), (2), (3) and the previous remark together, we conclude that there exists a real number $\delta>0$ such that

$$\|(y-x_{n_j})+(y-x_{n_{j+1}}),\ z\|\le\|y-x_{n_j},\ z\|+\|y-x_{n_{j+1}},\ z\|-\delta$$

or,

(4) $2\|y-\frac{1}{2}(x_{n_j}+x_{n_{j+1}}),\ z\|\le r_{n_j}+r_{n_{j+1}}-\delta$ for all $j\ge J_2$.

But $r_n\downarrow r$ implies the existence of $J>J_2$ such that

$$r_{n_j}-r\le\delta/4 \text{ for all } j>J.$$

Hence (4) becomes

$$\|y-\frac{1}{2}(x_{n_j}+x_{n_{j+1}}),\ z\|\le\frac{1}{2}\left(r+\frac{\delta}{4}+r+\frac{\delta}{4}-\delta\right)<r$$

Contradiction! since $\frac{1}{2}(x_{n_j}+x_{n_{j+1}})\in N$.

Therefore $\{x_n\}_{n=1}^{\infty}$ must be Cauchy and since N is closed and consequently complete, then there exists a unique $x_0\in N$ such that $x_n\to x_0$.

Next, we show uniqueness of x_0 with respect to (*). So suppose x_0 and x_0^* both satisfy (*) and $\|x_0-x_0^*,\ z\|>0$ for all $z\in E-V(x_0-x_0^*)$. Then

$$\frac{\|y-x_0,\ z\|}{r}=\frac{\|y-x_0^*,\ z\|}{r}=1$$

and,

$$\left\|\left(\frac{(y-x_0)}{r}-\frac{(y-x_0^*)}{r}\right),\ z\right\|=\frac{\|x_0-x_0^*,\ z\|}{r}\ge 2\varepsilon^*>0$$

from some $\varepsilon^*>0$. Then by definition (4.1), there exists $\eta<1$ such that

$$\frac{1}{r}\|(y-x_0)+(y-x_0^*),\ z\|\le 2\eta<2$$

or,

$$\|y-\frac{1}{2}(x_0+x_0^*),\ z\|<r$$

but $\frac{1}{2}(x_0+x^*{}_0)\in N$. Contradiction to (*)! Therefore x must be unique.

LEMMA 3.4. *Let E be a continuous 2-s.i.p. space which is uniformly convex and complete in its 2-norm and let N be a proper closed vector subspace of E. Then there is a vector z_0 which is normal to N.*

PROOF. Let $y\notin N$. Then by the last lemma, there exists a unique nonzero vector $x_0\in N$ such that

$$\|y-x,\ z\|=\inf\{\|y-x,\ z\| : x\in N\}.$$

Put $z_0=y-x_0$, then

$$\begin{aligned} \|z_0, z\| &= \|y - x_0, z\| \\ &\leq \|y - x, z\| \qquad \text{for all } x \in N \\ &= \|z_0 + x_0 - x, z\| \\ &= \|z_0 + t, z\| \qquad \text{for all } t \in N \end{aligned}$$

which implies that z_0 is normal to N [8, 3.3].

THEOREM 4.4. *In a continuous 2-s.i.p. space E which is uniformly convex and complete in its 2-norm, to every continuous linear 2-functional f defined on $E \times \{[z]\}$, $z \in E$, there exists a unique vector $y \in E$ such that*

$$f(x, z) = [x, y/z] \text{ for all } x \in E.$$

PROOF. If $f(x, z) = 0$ for all $x \in E$, then let $y = 0$.

If $f(x, z) \neq 0$ for some $x \in E$, then $N = \{x : f(x, z) = 0\}$ is a proper closed vector subspace of E.

Hence by last lemma, there exists a nonzero vector $y_0 \in E$ such that y_0 is normal to N.

When $x \in N$, we have

$$f(x, z) = [x, y/z] = 0 \text{ for } y = \alpha y_0, \ \alpha \in \boldsymbol{R}.$$

When $x = y_0$, we have

$$f(x, z) = [x, y/z] = f(y_0, z)$$

for the choice

$$y = \left\{ \frac{f(y_0, z)}{\|y_0, z\|^2} \right\} y_0, \ z \notin V(y_0)$$

Now since each $x \in E$ can be represented in the form $x = t + \lambda y_0$ where $t \in N$ and y_0 is normal to N and $\lambda = f(x, z)/f(y_0, z)$, we have for all $x \in E$

$$\begin{aligned} f(x, z) &= f(t + \lambda y_0, \ z) = f(t, z) + \lambda f(y_0, z) \\ &= [t, y/z] + \lambda [y_0, y/z] = [t + \lambda y_0, \ y/z] = [x, y/z]. \end{aligned}$$

Uniqueness: Suppose there exist vectors $y, y' \in E$, $y \neq y'$ such that

$$f(x, z) = [x, y/z] = [x, y'/z] \text{ for all } x \in E.$$

Then

$$[y, y/z] = [y, y'/z] \leq \|y, z\| \ \|y', z\|$$

or

$$\|y, z\|^2 \leq \|y, z\| \ \|y', z\|$$

$\Longrightarrow$

$$\|y, z\| \leq \|y', z\| \qquad (1)$$

Similarly, we can show that

$$\|y', z\| \leq \|y, z\| \tag{2}$$

and hence

$$\|y', z\| = \|y, z\| \tag{3}$$

From $\|y, z\|^2 = [y, y'/z]$, it follows that

$$\|y, z\| \ \|y', z\| = [y, y'/z].$$

Hence, by [8], 5.1, we have $y=y'$.

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