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LINEARLY COMPACT LEFT DUO RINGS AND THEIR STRUCTURE SPACES

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1. Introduction

Characterizations for linearly compact semisimple rings were obtained in [8, 9] and a similar characterization for linearly compact commutative rings was studied in [6]. In addition to these characterizations, and among other results, we will show that a linearly compact left duo ring is OM-semisimple if and only if it is rationally complete and biregular [1] or equivalently it is semisimple and its structure space is extremally disconnected.

2. Definitions and Notations

In this paper A will denote an associative ring with 1, not necessarily commutative, and all modules will be left unitary modules. Furthermore, all topological spaces are Hausdorff. A topological module is called linearly topologized if it admits a neighborhood base for zero consisting of submodules. By a linear variety in a module K we shall mean a coset of a submodule of K. A linearly topologized module K is linearly compact if every collection of closed linear varieties in K with the finite intersection property has a non-void intersection. A topological ring A is linearly compact in case it is a linearly compact A-module. A ring is a left duo ring if each left ideal of A is an ideal. Thus A is left duo if and only if, for each $x \in A$, $xA \subset Ax$. Right duo-ness has the obvious definition. For fundamental definitions and results related to rational extensions of rings, we refer to [3], [4] and [7]. The symbol $Q(A)$ will repr esent the rational completion of a ring A. Also, we use $\Omega(A)$ to denote the set of all maximal left ideals in A and, for each $a \in A$, $\Omega(a) \equiv \{M \mid M \in \Omega(A)$ and $a \notin M\}$. In case A is a left duo ring, $\Omega(A)$ can be endowed with the Stone-Zariski topology having the family $\{\Omega(a) \mid a \in A\}$ as a base. The space $\Omega(A)$ thus defined is called the structure space of A. It is well known in [1] that if A is a biregular ring then A is semisimple and $\Omega(A)$ is compact zero dimensional and A contains the characteristic function of any compact open subset of $\Omega(A)$. Now let Γ be a subset of Ω(A) with $\bigcap_{M \in \Gamma} M = (0)$, and let $B(\Gamma) = \{I \mid I$

is a left ideal in A and I contains a finite intersection of members of $\Gamma\}$. Then $B(\Gamma)$ is a base for a neighborhood system of 0 in A, and the linear topology on A thus generated by Γ will be termed the Γ-topology. Also, we use the notations $\Omega_{\circ}(A)$ and $\Omega_V(A)$ to symbolize the set of all open maximal left ideals, and the set of all rationally non-dense maximal left ideals, in A respectively. A linearly topologized ring A is OM-semisimple if $\bigcap_{M\in\Omega_{\circ}(A)} M=(0)$.

In what follows, a ring is said to be semisimple if its Jacobson radical is zero. We shall require the following theorems.

THEOREM 2.1 ([7], [4]). *If $\{A_i \mid i\in I\}$ is a family of rings, then* $Q(\prod_{i\in I}A_i)\cong\prod_{i\in I}Q(A_i)$.

THEOREM 2.2 ([9]). *A linearly compact ring A is semisimple if and only if every linearly compact A-module is injective.*

3. Main Results

We first prove the following lemma.

LEMMA 3.1. *Let A be a linearly compact semisimple left duo ring, and let M be an open maximal left ideal in A. If M' is any maximal left ideal in A with $M'\neq M$, then M and M' can be separated by basic open sets in the structure space of A.*

PROOF. We note that M is also closed. Thus M is a linearly compact A-submodule of A. Therefore M is injective by Theorem 2.2. It follows that there is a submodule L of A such that $M\oplus L=A$. Then there exist $m\in M$ and $b\in L$ such that $1=m+b$ with $b\neq 0$ and $b\notin M$. Let $a\in M\backslash M'$. Since $a=a(m+b)=am+ab$, we have $a-am=ab$, and hence $ab\in M$. But $ab\in L$. This implies that $ab=0$. Consequently $\Omega(a)\cap\Omega(b)=\Omega(ab)=\phi$ and $M\in\Omega(b)$ and $M'\in\Omega(a)$.

For a completely regular space X we use the symbol βX to denote the Stone-Čech compactification of X. We now state our main result.

THEOREM 3.2. *Let A be a linearly compact left duo ring. Then the following statements are equivalent.*

(1) *A is OM-semisimple.*

(2) $A\cong\prod A/M$ $(M\in\Omega_{\circ}(A))$.

(3) *A is rationally complete and biregular.*

(4) *A is semisimple and $\Omega(A)$ is extremally disconnected.*

(5) *A is semisimple and* $\Omega(A)=\beta\Omega_\circ(A)$.

PROOF. (1)$\Rightarrow$(2). A can be embedded into the product space $\prod A/M_i(M_i\in\Omega_\circ)$. Let κ be the embedding and $\hat{A}=\kappa(A)$. Since $\hat{A}$ is linearly compact, it is closed in the space $\prod A/M_i$ by [10]. Thus it suffices to show that $\hat{A}$ is topologically dense in $\prod A/M_i$. Note that the space A/M_i is discrete for each $M_i\in\Omega_\circ$. For elements $a_1, a_2, \cdots, a_n$ in A, we denote $\langle a_i\rangle\equiv a_i+M_i$ for each $i\in\{1, 2, \cdots, n\}$. Thus each $\langle a_i\rangle$ is open in A/M_i. Let $\prod W_i$ be a basic open set in $\prod A/M_i$ where $W_i=\langle a_i\rangle$ for $i\in\{1, 2, \cdots, n\}$ and $W_i=A/M_i$ for all $i\notin\{1, 2, \cdots, n\}$. By Lemma 3.1 there exist nonzero elements $c_1, c_2, \cdots, c_n$ in A such that $M_i\in\Omega(c_i)$ for each $i\in\{1, 2, \cdots, n\}$ and $\Omega(c_i)\cap\Omega(c_j)=\phi$ for distinct i and j in $\{1, 2, \cdots, n\}$. Hence we have $c_i\in M_j$ for $i\neq j$. Since M_i is maximal for each i, there exist elements $b_1, b_2, \cdots, b_n$ in A such that $b_ic_i+M_i=1+M_i$ for each $i\in\{1, 2, \cdots, n\}$. Now let $a=a_1b_1c_1+a_2b_2c_2+\cdots+a_ib_ic_i+\cdots+a_nb_nc_n$. Then $a\in A$ and $a+M_i=(a_1b_1c_1+a_2b_2c_2+\cdots+a_ib_ic_i+\cdots+a_nb_nc_n)+M_i=a_ib_ic_i+M_i=a_i(b_ic_i+M_i)=a_i+M_i$ for each $i\in\{1, 2, \cdots, n\}$. That is $a+M_i=\langle a_i\rangle$. Also note that $a+M_p\in A/M_p$ for $p\notin\{1, 2, \cdots, n\}$. This implies that $\hat{a}\in\prod W_i$ where $\hat{a}=\kappa(a)$. $\hat{A}$ is topologically dense in $\prod A/M_i$ $(M_i\in\Omega_\circ)$.

(2)$\Rightarrow$(3). Let $a\in A$. We claim that $(a)=(e)$ for an idempotent e in the center of A where (a) denotes the two sided ideal generated by a. Let $M_i\in\Omega_\circ$. If $\hat{a}(M_i)=a+M_i\neq 0$, then there exists an $x_i\in A$ such that $x_ia+M_i=1+M_i$. Define a function $\hat{x}:\Omega_\circ\to\cup A/M_i$ $(M_i\in\Omega_\circ)$ by

$$\hat{x}(M_i)=\begin{cases}\hat{x}_i(M_i) & \text{for } M_i \text{ if } \hat{a}(M_i)\neq 0\\ 0 & \text{for } M_i \text{ if } \hat{a}(M_i)=0.\end{cases}$$

Let $x\in\kappa^{-1}(\hat{x})$ and $e=xa$. Since $(\hat{x}\hat{a})(M_i)=\hat{x}(M_i)\hat{a}(M_i)=\hat{x}_i(M_i)\hat{a}(M_i)=(\hat{x}_i\hat{a})(M_i)=x_ia+M_i=\hat{1}(M_i)$ for each M_i with $\hat{a}(M_i)\neq 0$, e is an idempotent. Also for $b\in A$, we have $be=eb$, and clearly $(a)=(e)$. Thus A is biregular. Using Theorem 2.1, we have $Q(A)\cong\prod_{M\in\Omega_\circ}Q(A/M)\cong\prod_{M\in\Omega_\circ}A/M\cong A$. Thus A is rationally complete.

(3)$\Rightarrow$(4). Let $\mathring{A}$ be the set of all idempotents of A. Then by [5], $\Omega(\mathring{A})\cong\Omega(A)$. Since A is rationally complete, $\mathring{A}$ is also complete [3]. Thus $\Omega(\mathring{A})$ is extremally disconnected and so is $\Omega(A)$. A is semismple as stated earlier.

(4)$\Rightarrow$(5). It is well known that a compact Hausdorff space is extremally disconnected if and only if it is a Stone-Čech compactification of every dense subspace of the space. Since a linearly compact semisimple ring is OM-semisimple (see [6]), $\Omega_\circ(A)$ is dense in $\Omega(A)$. Thus $\Omega(A)=\beta\Omega_\circ(A)$.

(5)⇒(1). If $\Omega(A)=\beta\Omega_\circ(A)$, then $\Omega_\circ(A)$ is dense in $\Omega(A)$. Hence $\cap\Omega_\circ(A)=\cap\Omega(A)$. But A is semisimple, and thus $\cap\Omega_\circ(A)=(0)$.

4. Applications

LEMMA 4.1. *If A is semisimple commutative, then a maximal ideal M is not rationally dense in A if and only if $\{M\}=\Omega(a)$ for some $a\neq 0$ in A.*

PROOF. Let M be rationally non-dense in A. Then there exists $a\neq 0$ in A with $aM=0$, i.e., $am=0$ for all m in M. Hence $\Omega(a)\cap\Omega(m)=\phi$ for each $m\in M$. Let $Z(m)=\Omega(A)/\Omega(m)$. Then $\Omega(a)\subset\bigcap_{m\in M}Z(m)$. But $\bigcap_{m\in M}Z(m)$ contains at most one element. Since $\Omega(a)\neq\phi$, we have $\Omega(a)=\{M\}$. Conversely, if $\{M\}=\Omega(a)$ for some $a\neq 0$, then $a\in M'$ for all $M'\in\Omega(A)$ with $M'\neq M$. We note that $aM\subset(\bigcap_{\substack{M'\in\Omega(A)\\ M'\neq M}}M')\cap M=(0)$. Thus $aM=0$. It follows that M is not rationally dense in A.

PROPOSITION 4.2. *A linearly topologized Boolean ring is linearly compact if and only if it is compact.*

PROOF. If A is a linearly compact Boolean ring, then for each open maximal ideal M, A/M is compact. By Theorem 3.2. A is compact. The converse is clear.

PROPOSITION 4.3. *A Boolean ring A is complete and atomic if and only if it is compact with respect to $\Omega_r(A)$-topology.*

PROOF. If A is complete and atomic, then the space $\Omega(A)$ is extremally disconnected and it contains a dense subset Σ of isolated points in $\Omega(A)$. By Lemma 4.1 the isolated points in $\Omega(A)$ are precisely the rationally nondense maximal ideals in A. Thus $\Sigma=\Omega_r(A)$. Note that $\cap\{M|M\in\Omega_r(A)\}=(0)$. Hence A is a linearly topologized ring endowed with the $\Omega_r(A)$-topology. Also note that every element of $\Omega_r(A)$ is open. Furthermore A can be considered as a subring of $\prod A/M$ $(M\in\Omega_r(A))$. Now take an element a in $\prod A/M$. Let $S=\{M|M\in\Omega_r(A)$ and $a(M)=1\}$ and $Z=\{M|M\in\Omega_r(A)$ and $a(M)=0\}$. Then both S and Z are open and disjoint subsets of $\Omega(A)$. Since $\Omega(A)$ is extremally disconnected, we have $\bar{S}\cap\bar{Z}=\phi$, where "$-$" denotes the closure operator. Thus there exists a characteristic function a^* in A such that $a^*(M)=1$ for all M in $\bar{S}$ and $a^*(M)=0$ for $M\notin\bar{S}$. Hence $a^*=a$. It follows that $A=\prod A/M(M\in\Omega_r(A))$ and A is compact.

Conversely, if A is compact with respect to the $\Omega_{\gamma}(A)$-topology, then it is linearly compact with respect to the same topology. By Theorem 3.2 A is complete. Since $\Omega_{\gamma}(A)$ is a dense subset of isolated points, the Boolean ring A is atomic.

PROPOSITION 4.4. *Let X be a space of nonmeasurable cardinal* [2]. *Then $C(X)$, the ring of real-valued continuous functions on X, is linearly compact if and only if the space X is discrete.*

PROOF. If $C(X)$ is linearly compact, then by Theorem 3.2 it is a regular ring. Hence X is a P-space [2]. Also by Theorem 3.2 βX is extremally disconnected, and so is X. Thus X is discrete. The converse is evident.

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