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ON GENERALIZED SOLVABLE AND NILPOTENT GROUPS

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0. Introduction

A necessary and sufficient condition is given on an integer so that a group having order n should possess a certain structural property. Necessary and sufficient conditions are also posed on a group possessing a certain invariant series to be upper nilpotent. Investigation is extended to give a condition for a group having a certain invariant system to be SN-group, SI-group, Z-group or an upper nilpotent group.

In this paper we adopt the notions used in Kuros' monograph [5] as well as those cited in the second author's paper [3]. We refer the reader to both works for unexplained terminology used here without reference.

1. Nilpotency properties of groups of order n and upper nilpotency properties of groups composed of abelian groups of finite rank.

DEFINITION. A group for which every finitely generated subgroup has an order dividing at least one integer $n\in\Gamma$, $n\neq 1$ is called a Γ-group, Γ being a set of positive integers.

LEMMA 1. (G. Pazderski, see [7]) *All groups of order* $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ ($p_1, \cdots, p_r$ *different primes,* r, $\alpha_1, \cdots, \alpha_r$ *integers) are nilpotent if and only if* $p_i \nmid p_j^{\nu}-1$, $1\leq\nu\leq\alpha_j$, $i, j=1, \cdots, r$.

LEMMA 2. *All groups of order* $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ *are nilpotent of class at most* c *if and only if* $p_i \nmid p_j^{\nu}-1$, $1\leq\nu\leq\alpha_j$, $i,j=1,\cdots,r$, *and* $\alpha_i\leq c+1$.

PROOF. If n has the property stated, then every group G of order n is nilpotent (lemma 1), i.e. G is the direct product of its Sylow subgroups P_1, $\cdots$, P_r where P_i is a group of order $p_i^{\alpha_i}$ ($i=1, \cdots, r$). If P is a group of order p^{α} ($\alpha\geq 2$ integer, p prime) then is nilpotent of class at most $\alpha-1$ because $|p:p'|\geq p^2$. Therefore the class of nilpotency of G is at most $c=\max(c(P_1)$, $\cdots$, $c(P_r))\leq\max(\alpha_1-1, \cdots, \alpha_r-1)\leq c$ unless $\alpha_1=\cdots=\alpha_r=1$; but in the last case

G is commutative, so $c(G) \leq c$.

Conversely, if all groups of order n are nilpotent of class at most c, then n satisfies the stated properties (lemma 1) with the exception $\alpha_i \leqq c+1$; but it also holds, since for every $\alpha \geq 2$ integer and p prime, there is a p-group of nilpotency class $\alpha - 1$.

THEOREM 3. *If Γ is a set of integers such that all Γ-groups are locally nilpotent, then all Γ-groups are nilpotent or p-groups.*

PROOF. For a fixed $n \in \Gamma$, all groups of order n will be Γ-groups. Since all Γ-groups are locally nilpotent, then all groups of order n are nilpotent. Hence every $n \in \Gamma$ satisfies the condition of lemma 1. Let H be finitely generated subgroup of a Γ-group G. If G is not a p-group, then we can find $a, b \in G$ with different prime orders p_0 and q_0 respectively, and assume that $p_0 < q_0$. The group $\{a, b, H\}$ is a finite subgroup of G and so its order divides a number $n \in \Gamma$, and so p_0, q_0 and $|H|$ are divisors of n. Hence for each $p^\alpha /\!/ |H|$ we have $q_0 + p_0^\alpha - 1$; therefore either $\alpha < p_0 - 1$ or $\alpha < q_0 - 1$. By lemma 2, the class a nilpotency of H is certainly smaller than q_0. Since this holds for every finite subgroup of the locally finite group G, we have that G is nilpotent.

THEOREM 4. *If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $r_{p_i} \geq 1$ $(i=1, \cdots, k)$ are given integers, then all groups of order n the Sylow p_i-subgroups of which are generated by r_{p_i} elements are nilpotent if and only if $p_i + p_j^\nu - 1$ holds for all $1 \leqq \nu \leqq r_{p_i}$ $(i, j=1, \cdots, k)$.*

PROOF. Let G be a group of order n satisfies the condition of the theorem. If n is even, then no odd prime can divide n, so in this case G is a 2-group, and so it is nilpotent. If n is odd, then G is of odd order and so it is solvable, and so G has a nontrivial abelian normal subgroup the Sylow p-subgroups of which are normal is G. Let $P \neq 1$ be a maximal normal p-subgroup of G, then G/P has no $\neq 1$ normal p-subgroup. (By induction on the order of G we have G/P is nilpotent). Therefore $(|P|, |G/P|) = 1$, and so there exists a subgroup H of G such that $PH = G$, $P \cap H = 1$. Consider the homomorphic mapping $\bar{x} \xrightarrow{\varphi_h} h^{-1}xh$ where $x \in \bar{x}$, $\bar{x} \in P/\phi(P)$, $h \in H$ and $\phi(P)$ is the Frattini subgroup of P. Thus the homomorphism $h \to \varphi_h$ maps H into the automorphism group $\mathrm{Aut}(P/\phi(P))$ of $P/\phi(P)$. Since the number of generators of $P/\phi(P)$ does not exceed that of P, then $|P/\phi(P)| = p^d$ where $1 \leq d \leq r_p$. Suppose that q is a prime such that $q/0(\varphi_h)$; then $q/|\text{Aut}\ (P/\phi(P))|$. Since $0(\varphi_h)/0(h)$, then $q/0(\text{h})$; and so $q/|G|$. Since $P/\phi(P)$ is elementary abelian, then $|\text{Aut}\ P/\phi(P)|=(p^d-1)\cdots(p^d-p^{d-1})$. Then $q/p^{\frac{d(d-1)}{2}}(p^d-1)\cdots(p-1)$. Therefore there exists $1\leq\nu\leq d\leq r_p$ such that $q/p^{\nu}-1$ which is a contradiction with $q/|G|$. Hence no prime $q/0(\varphi_h)$ can exist, and so $0(\varphi_h)=1$; which means that every element $h\in H$ transforms $P/\phi(P)$ identically, then H is normal in G. Thus G is the direct product of P and H, but H is nilpotent by induction, so G is nilpotent.

To prove the converse, let p_ρ, p_m be different primes among $p_1, \cdots, p_k$ such that $p_m/p_\rho^\gamma-1$ for some integer $1\leq\gamma\leq\gamma_{p_\rho}$. Consider the field F of charateristic p_ρ with p_ρ^γ elements and denote F^x the multiplicative group $F/\{0\}$ and F^+ the additive group of F which is elementary abelian. Let Q be a subgroup of F^x having p_m elements. For an element $a\in Q$ the homomorphic mapping $x\xrightarrow{\theta_a}xa$ (where $x\in F^+$) is an endomorphism of F^+. Therefore the homomorphic mapping $a\xrightarrow{\rho}\theta_a$ maps Q homomorphically into Aut F^+. It is clear that ρ is not the trivial representation of Q over the vector space F^+. By the homomorphic mapping $x\xrightarrow{\theta_a}xa=x^a$ for $a\in Q$ and $x\in F^+$, the symbols (a, x) form a group G_1 under the product rule $(a_1, x_1)\ (a_2, x_2)=(a_1a_2, x_1^{a_2}x_2)$ where $a_1, a_2\in Q$ and $x_1, x_2\in F^+$. G_1 is a semidirect product of two subgroups K_1 and Q_1 isomorphic to F^+ and Q respectively ; and so $|G_1|=p_\rho^\gamma\, p_m$. This group G_1 is not nilpotent, for the representation of its subgroup Q_1 over the normal subgroup K_1 is nontrivial. Now, let G_2 be the direct product of G_1 and the cyclic group of order $n/p_\rho^\gamma\, p_m$, then the order of G_2 is the given integer n, and moreover it is not nilpotent.

LEMMA 5. (see [4]) *A super solvable torsion group G is upper nilpotent if for any two elements $a, b\in G$, if $p/0(a)$ and $q/0(b)$, then $p\nmid q-1$. Conversely, if $p/q-1$, then G must not be upper nilpotent.*

THEOREM 6. *A supersolvable group G is upper nilpotent if for any two elements $a, b\in\bar{G}$ of any factor group $\bar{G}$ of G, if $p=0(a)$ and $q=0(b)$, then $p\nmid q-1$. Conversely, if $p/q-1$, then G must not be upper nilpotent.*

PROOF. The supersolvable group G has an ascending chain of normal subgroups with cyclie factors. If G in not torsion then one can associate to this chain a factor group G/H with an infinite cyclic normal subgroup. Therefore there is $H \triangleleft B \triangleleft A \triangleleft G$ such that A/H is an infinite cyclic group and $|A:B|=6$ which implies that there are elements of order 2 and 3 in G/B, contradicting our condition that $p \nmid q-1$. Hence G is torsion and by lemma 5 it should be upper nilpotent as well. The converse statement follows at once from lemma 5.

LEMMA 6. (see [4]) *A solvable torsion group G of rank at most r is upper nilpotent if for any two elements $a, b \in G$, if $p/0(a)$ and $q/0(b)$ then $p \nmid q^{\nu}-1$, $1 \leq \nu \leq r$. Conversely, if $p/q^{\gamma}-1$ for some $1 \leq \gamma \leq r$, then G must not be upper nilpotent.*

THEOREM 7. *A solvable group G of rank at most r is upper nilpotent if for any two elements $a, b \in \bar{G}$ of any factor group $\bar{G}$ of G, if $p=0(a)$ and $q=0(b)$, then $p \nmid q^{\nu}-1$, $1 \leq \nu \leq r$. Conversely, if $p/q^{\gamma}-1$ for some $1 \leq \gamma \leq r$, then G must not be upper nilpotent.*

PROOF. If the solvable group G of rank at most r is not a torsion group then it has at least one factor group G/H which contains an infinite abelian normal subgroup $A=K/H$ with at most r generators. If $T=K_1/H$ is the periodic part of A, then the factor group $B \approx A/T \approx K/K_1$ is the direct product of at most r infinite cyclic subgroups. Then $B^6=K_2/K_1$ is such a characteristic subgroup of B with $|B:B^6|$ is divisible by 6, in other words $|(K/K_1):(K_2/K_1)| \approx |K:K_2|$ is divisible by 6 and so 6 divides the order of a subgroup of the factor group G/K_2. Therefore the factor group G/K_2 has elements of orders 2, 3 which contradicts the fact that $p \nmid q-1$. Hence G is a torsion group and according to lemma 6 G should be upper nilpotent. For the converse of theorem an appeal to lemma gives directly the required result and this completes the proof.

2. Similar results using the concept of invariant systems

We start by proving the following result.

THEOREM 8. *If in a group G every chief factor is nilpotent of class at most c, then is $\overline{SN}$-group.*

PROOF. Let G be a group with the prescribed property and let B a subgroup of G.

Firstly, if B is simple, then B is nilpotent of class at most c. So every maximal subgroup of B is normal and hence G is $\tilde{N}$-group and so it must be an $\overline{SN}$-group.

Secondly, if B is not simple, then there is at least a maximal normal proper subgroup N of B and B/N is simple. Consider a maximal proper subgroup A of B such that $A \supset N$. Since N is normal of B, then N is normal of A, and so A/N is a subgroup of B/N which contradicts the condition that B/N is simple. Then in every subgroup B of G, every maximal proper subgroup A of B is normal in B, and so G is an $\tilde{N}$-group, and since every $\tilde{N}$-group is an $\overline{SN}$-group G is certainly an $\overline{SN}$-group.

THEOREM 9. *If any chief factor of a given group is cyclic, then the group itself is a Z-group.*

PROOF. Let a group G is such that any chief factor is cyclic, and let B be a subgroup of G.

Firstly, if B is simple, then B is cyclic, and so every proper maximal subgroup A of B is normal in B. But this is a condition for G to be an $\tilde{N}$-group (see [4]).

Secondly, if B is not simple, then there is at least a maximal normal proper subgroup N of B such that B/N is simple. Let A be a maximal proper subgroup of B such that $A \supset N$. Since N is normal of B, then N is normal of A. So $A/N=1$ is a subgroup of B/N which is a contradiction with the condition that B/N is simple. In other words in every subgroup B of G, every maximal proper subgroup A of B is normal in B. Then G is an $\tilde{N}$-group having invariant system with cyclic factor. But this condition forces that G should be Z-group.

THEOREM 10. *If every proper subgroup of a given group is cyclic, then the group itself is upper nilpotent.*

PROOF. Let a group G is such that any proper subgroup is cyclic and let $B \neq 1$ is a subgroup of G. Then B is cyclic and every maximal proper subgroup A of B is normal in B. So G should be an $\tilde{N}$-group. Since G is an $\tilde{N}$-group such that any proper subgroup is cyclic, then there is a maximal subgroup H of G which is cyclic. So H has an ascending invariant series with cyclic factor. Therefore G has an acending invariant series with cyclic factors and this means that G is an upper nilpotent (see [4]).

THEOREM 11. *If every proper subgroup of a given group G is abelian then G is an SN*-group.*

PROOF. Let G be a group such that any of its proper subgroups is abelian. If B is a subgroup of G, then B is abelian and every maximal proper subgroup A of B is normal in B, and so G is $\tilde{N}$-group. Since every subgroup of G is abelian, then there is a maximal subgroup H of G that possesses an ascending invariant series with abelian factor. Therefore G has an ascending normal series with abelian factors. Hence G is SN^*-group.

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