Kyungpook Math. J.
Volume 26, Number 1
June, 1986

ROTATIONAL MENDELSOHN TRIPLE SYSTEMS

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1. Introduction

A *cyclic triple* is a set T of three ordered pairs such that an element occurs as a first coordinate of an ordered pair in T if and only if it occurs as a second coordinate of an ordered pair in T. We will denote the cyclic triple $\{(a,b),\ (b,c),\ (c,a)\}$ by (a,b,c), (b,c,a) or (c,a,b). A *Mendelsohn triple system* MTS(v) of order v is a v-set and B is a collection of cyclic triples of elements of V (called blocks) such that every ordered pair of distinct elements of V belongs to exactly one block. It is well-known [5] that a MTS(v) exists if and only if $v\equiv 0$ or 1 (mod 3) and $v\neq 6$. An *automorphism* of a MTS(v) (V,B) is a permutation α on V which preserves B. A MTS(v) is said to be *k-rotational* if it admits an automorphism α consisting of a single fixed element and exactly k $\frac{(v-1)}{k}$-cycles; and α is called a *k-rotational automorphism*. If a permutation α of degree v consists of a single v-cycle, then a MTS(v) admitting α as its automorphism is called *cyclic*. It is shown by Colbourn and Colbourn [1] that a cyclic MTS(v) exists if and only if $v\equiv 1$ or 3 (mod 6) and $v\neq 9$.

In this paper, we obtain a necessary and sufficient condition for the existence of 1-rotational MTS(v).

A *Steiner triple system* STS(v) of order v is a pair (V,B) where V is a v-set and B is a collection of 3-subsets of V (called triples) such that every 2-subset of V belongs to exactly one triple. It is well-known that a STS(v) exists if and only if $v\equiv 1$ or 3 (mod 6), and Peltesohn [6] first shows that a cyclic STS(v) exists if and only if $v\equiv 1$ or 3 (mod 6) and $v\neq 9$.

An (A,k)-*system* is a set of ordered pairs $\{(a_r,b_r)\,|\,r=1,2,\cdots,k\}$ such that $b_r-a_r=r$ for $r=1,2,\cdots,k$, and $\cup_{r=1}^{k}\{a_r,b_r\}=\{1,2,\cdots,2k\}$. It is well-known [see 7] that an (A,k)-system exists if and only if $v\equiv 0$ or 1 (mod 4).

2. 1-Rotational Mendelsohn Triple Systems

Let Z denote the set of all integers and let Z_v be the group of residue classes of Z modulo v. Throughout, we assume that the set of elements of our 1-rotational MTS(v) is $V=Z_{v-1}\cup\{\infty\}$ and the corresponding 1-rotational automorphism is $\alpha=(\infty)\ (0\ 1\cdots v-2)$.

For each element $a\in Z_{v-1}$, define $a\pm\infty=\pm\infty$. We can associate each cyclic triple (a,b,c) of elements of $Z_{v-1}\cup\{\infty\}$ with a *difference triple* (x,y,z) where $x\equiv b-a$, $y\equiv c-b$, and $z\equiv a-c$ (mod $v-1$). Note that the cyclically shifted cyclic triples of a cyclic triple are equivalent, i.e. they contain the same ordered pairs and hence the cyclically shifted difference triples of a difference triple are equivalent, i.e. they correspond the same cyclic triples. Also, note that difference triples are of two types: either an ordered triple (x,y,z) for which $x+y+z\equiv 0$ (mod $v-1$) or $(x,\infty,-\infty)$ and $x\neq\infty$. An *orbit* of a 1-rotational MTS(v) is a collection of all blocks with the same difference triple. Thus, each orbit of a 1-rotational MTS(v) corresponds a unique difference triple, and conversely. A collection of starter blocks of a 1-rotational MTS(v) is a collection of blocks which are taken exactly one from each orbit.

Applying Heffter's [4] two so-called *difference problems* (see [2] for a detailed description), a 1-rotational MTS(v) for $v\equiv 0$ (mod 3) is equivalent to a partitioning of the set $\{1,2,\cdots,v-2\}\setminus\{k\}$ for some $1\leq k\leq v-2$ into difference triples; here, a difference triple is an ordered triple (x,y,z) for which $x+y+z\equiv 0$ (mod $v-1$). When $v\equiv 1$ (mod 3), a 1-rotational MTS(v) is equivalent to a partitioning of $\{1,2,\cdots,v-2\}\setminus\{k,t\}$ for some $1\leq k\leq v-2$, $t=\frac{v-1}{3}$ or $\frac{2(v-1)}{3}$ and $t\neq k$ into difference triples. These simple observations enable us to prove the follwing necessary condition.

LEMMA 2.1. *If there exists a 1-rotational MTS(v), then* $v\equiv 1, 3$ *or* 4 (mod 6).

PROOF First of all, we have $v\equiv 0$ or 1 (mod 3) and $v\neq 6$, since this is the spectrum for MTS(v). In case $v\equiv 0$ (mod 6) and $v\neq 6$, the existence of a 1-rotational MTS(v) is equivalent to a partitioning of the set $\{1,2,\cdots,v-2\}\setminus\{k\}$ for some $1\leq k\leq v-2$ into difference triples (x,y,z) for which $x+y+z\equiv 0$ (mod $v-1$). Since $v-1$ divides the sum of the differences in each difference triple, it divides the sum of all differences being partitioned into difference triples. Thus, $v-1$ divides the sum of the integers 1 through $v-2$ except exactly one integer, i.e. $\frac{(v-2)(v-1)}{2}-k\equiv 0$ (mod $v-1$) for some $1\leq k\leq v-2$, but there is no such an integer k. Hence there exists no 1-rotational MTS(v) for $v\equiv 0$ (mod 6).

LEMMA 2.2. [3]. *There exists no* 1-*rotational* $MTS(10)$.

LEMMA 2.3. *If* $v\equiv 4 \pmod 6$ *and* $v\neq 10$, *then there exists a* 1-*rotational* $MTS(v)$.

PROOF. Let $v=6t+4$ and $t\neq 1$. Then

$\{(0,\infty,2t+1),\ (0,2t+1,4t+2)\}$,

$\{(a,b,c),\ (a,c,b)\mid \{a,b,c\}\in C\}$

where $C\cup\{\{0,2t+1,4t+2\}\}$ is a collection of starter triples of a cyclic STS $(6t+3)$,

are a collection of starter blocks of a 1-rotational MTS$(6t+4)$, $t\neq 1$.

LEMMA 2.4. *If* $v\equiv 7$ *or* 13 (mod 18), *then there exists a* 1-*rotational* $MTS(v)$.

PROOF. Let $v=6t+1$ and $t\equiv 1$ or 2 (mod 3). Then

$\{(0,\infty,t),\ (0,4t,2t)\}$,

$\{(0,3r,2t-3+6r)\mid r=1,2,\cdots,t\}$,

$\{(0,3r,6r-4t)\mid r=t+1,t+2,\cdots,2t-1\}\ (t>1)$

are a collection of starter blocks of a 1-rotational MTS$(6t+1)$ where $t\equiv 1$ or 2 (mod 3).

LEMMA 2.5. *If* $v\equiv 1$ (mod 18), *then there exists a* 1-*rotational* $MTS(v)$.

PROOF. Let $v=6t+1$ and $t\equiv 0$ (mod 3). Then

$\{(\infty,0,t),\ (0,2t,4t)\}$,

$\{(0,3t+1-r,r)\mid r=1,2,\cdots,t\}$,

$\{(0,r,7t-r)\mid r=t+1,t+2,\cdots,2t-1\}$

are a collection of starter blocks of a 1-rotational MTS$(6t+1)$ where $t\equiv 0$ (mod 3).

LEMMA 2.6. If $v\equiv 3$ or 9 (mod 24), then there exists a 1-rotational MTS(v).

PROOF. Let $v=6t+3$ and $t\equiv 0$ or 1 (mod 4), Then

$\{(\infty,0,3t+1)\}$,

$\{(0,r,b_r+t),\ (0,b_r+t,r)\mid r=1,2,\cdots,t\}$

where $\{(a_r,b_r)\mid r=1,2,\cdots,t\}$ is an (A,t)-system,

are a collection of starter blocks of a 1-rotational MTS$(6t+3)$ where $t\equiv 0$ or 1 (mod 4).

LEMMA 2.7. *If* $v\equiv 15$ *or* 21 (mod 24), *then there exists a* 1*-rotational* $MTS(v)$.

PROOF. Let $v=6t+3$ and $t\equiv 2$ or 3 (mod 4). Then

$\{\infty, 0, 3t+1)\}$,

$\{(0, r, 3t+1-r),\ (0, 5t+2-r, r) \mid r=1, 2, \cdots, t\}$

are a collection of starter blocs of a 1-rotational MTS $(6t+3)$ where $t\equiv 2$ or 3 (mod 4).

Summarizing, we have

THEOREM 2.8. *A* 1*-rotational* $MTS(v)$ *exists if and only if* $v\equiv 1, 3$ *or* 4 (mod 6) *and* $v\neq 10$.

3. Concluding Remarks

Note that a 1-rotational MTS(v) exists for all admissible orders v which are the spectrum for the existence of a MTS(v), except for $v\equiv 0$ (mod 6) and $v=10$. If $v\equiv 0$ (mod 6) and $v\neq(6t+1)(6k-1)+1$, then $v-1$ is a prime number. Thus, for the orders $v\equiv 0$ (mod 6) and $v\neq(6t+1)(6k-1)+1$, only $(v-1)$-rotational MTS(v) are considered; clearly such systems exist as their existence trivially follows from the existence of MTS (since the $(v-1)$-rotational automorphism is exactly the identity automorphism). In addition, a 3-rotational MTS(10) exists. For example, $(\infty, 1, 0)$, $(\infty, 4, 3)$, $(\infty, 7, 6)$, $(0, 1, 3)$, $(3, 4, 6)$, $(0, 6, 7)$, $(0, 4, 8)$, $(0, 8, 4)$, $(0, 3, 6)$, and $(0, 7, 5)$ with $\alpha=(\infty)\ (0\ 1\ 2)\ (3\ 4\ 5)\ (6\ 7\ 8)$ are a collection of starter blocks of a 3-rotational MTS(10). Therefore, the only unsettled problem for the existence of rotational MTS is: If $v=(6t+1)(6k-1)+1$, do there exist a $(6t+1)-$and a $(6k-1)-$rotational, respectively, ?

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