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Unbounded Scalar Operators on Banach Lattices

by

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Abstract

We show that a (possibly unbounded) linear operator, T, is scalar on the real line (spectral operator of scalar type, with real spectrum) if and only if (iT) generates a uniformly bounded semigroup and $(1-iT)(1+iT)^{-1}$ is scalar on the unit circle. T is scalar on $[0,\infty)$ if and only if T generates a uniformly bounded semigroup and $(1+T)^{-1}$ is scalar on $[0,1)$. By analogy with these results, we define C^0-scalar, on the real line, or $[0,\infty)$, for an unbounded operator. We show that a generator of a positive-definite group is C^0-scalar on the real line, and a generator of a completely monotone semigroup is C^0-scalar on $[0,\infty)$. We give sufficient conditions for a closed operator, T, to generate a positive-definite group: the sequence $<\phi(T^n x)>_{n=0}^{\infty}$ must equal the moments of a positive measure on the real line, for sufficiently many positive ϕ in X^*, x in X. If the measures are supported on $[0,\infty)$, then T generates a completely monotone semigroup. On a reflexive Banach lattice, these conditions are also necessary, and are equivalent to T being scalar, with positive projection-valued measure. T generates a completely monotone semigroup if and only if T is positive and m-dispersive and generates a bounded holomorphic semigroup.

1. Introduction

Scalar operators (spectral operators of scalar type) are a generalization, to arbitrary Banach spaces, of self-adjoint operators on a Hilbert space. The theory of spectral operators was initiated by Dunford ([6]).

There is a shortage of usable sufficient conditions on an operator that guarantee it will be scalar (unless we're on a Hilbert space). The following result is due to Bark-

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son ([3]): a bounded operator, T, on a reflexive Banach space, is scalar on the real line if and only if there exists an equivalent norm with respect to which T^n is Hermitian, for all n. In [7] and [8], Kantorovitz gives sufficient conditions for a (possibly unbounded) generator on a reflexive Banach space to be scalar.

When T is a bounded scalar operator, it has a *functional calculus*, that is, a continuous algebra homomorphism, Λ, from the space of bounded Borel measurable functions, defined on the spectrum of T, into the space of bounded operators, with $\Lambda f_0 = I$, $\Lambda f_1 = T$, where $f_0(t) \equiv 1$, $f_1(t) \equiv t$. For example, when T is multiplication by a function h, on a space of functions, then Λf is multiplication by f composed with h.

A bounded C^0-scalar operator is one that has a functional calculus defined for functions continuous on its spectrum. We extend the definition to unbounded operators (Definition 3.2). The previous paragraph shows that any scalar operator is C^0-scalar. When we're on a reflexive space, the converse is also true. An example of a C^0-scalar operator that is not scalar is multiplication by f_1, on $C[0,1]$.

Many results in operator theory involve the rate of growth of $\|T^n x\|$, as n goes to ∞. When x is an *analytic vector* for T, that is, $\sum^{\infty} \frac{s^n}{n!}\|T^n x\|$ is finite, for some $s>0$, then $e^{isT}x = \sum \frac{(is)^n}{n!} T^n x$ defines a 1-parameter group generated by T. When T is a symmetric operator on a Hilbert space, the group is unitary. Nelson, in 1959, ([10]) first showed that, if T is symmetric and has a dense set of analytic elements, then the closure of T is self-adjoint.

A short proof of Nelson's result (see [4]) uses merely the fact, well-known from complex analysis, that if f is an analytic function on the real line, and $f^{(k)}(0)=0$, for all natural number k, then $f(t)=0$, for all real t. A generalization of analytic is *quasi-analytic.* A sequence of positive numbers $\{M_n\}_{n=0}^{\infty}$ is quasi-analytic if, whenever $f^{(k)}(0)=0$, $\|f^{(k)}\|_\infty \leq M_k$, for all $k \in N$, then $f(t)=0$ for all $t \in R$. The Denjoy-Carleman theorem ([13], p. 412) says that $\{M_n\}_{n=0}^{\infty}$ is quasi-analytic if and only if $\sum_{n=0}^{\infty} \left(\inf_{k \leq n} M_k^{-1/k} \right) = \infty$.

Quasi-analytic vectors were introduced by Nussbaum [11]. A vector $x \in C^\infty(A)$ is a *quasi-analytic vector for* A if $\{\|A^n x\|\}_{n=0}^{\infty}$ is a quasi-analytic sequence. Nussbaum showed that when A is symetric and closed on a Hilbert space, and has a dense set of quasi-analytic vectors, then A is self-adjoint. Chernoff [4], showed that if a closed operator on a Banach space has an m-accretive extension (possibly on a larger space) and a dense set of quasi-analytic vectors, then the operator is m-accretive. It is not known if every accretive operator has an m-accretive extension (possibly on a larger space).

A sequence satisfies a *Stieltjes* growth condition if $\sum_{n=0}^{\infty}\left(\inf_{k\le n} M_k^{-1/2k}\right)=\infty$.

The vector $x\in C^{\infty}(A)$ is a *Stieltjes vector* if $\{\|A^n x\|\}$ satisfies a Stieltjes growth condition. Chernoff [4] showed that if A is closed and has an extension that generates a bounded holomorphic semigroup, and has a dense set of Stieltjes vectors, then A generates a bounded holomorphic semigroup.

The *moment problem*, from classical analysis, asks the following question: for which sequences $\{a_n\}_{n=0}^{\infty}$ does there exit a positive measure u such that $a_n=\int_R t^n du(t)$ for all n? We will present preliminary material on this in Section 3.

The moment problem is closely connected to the spectral theorem. In [12], p. 145, the spectral theorem is used to prove that a sequence is positive-definite if and only if it equals the moments of a positive measure on the real line. Uniqueness, when the sequence is quasi-analytic, follows from the fact that a symmetric operator with a dense set of quasi-analytic vectors is self-adjoint. Proofs of the spectral theorem, using moment theory, have been given, for example, by Akhiezer ([1]), and Schaefer ([14]).

This paper uses moment theory to derive sufficient conditions for a (possibly unbounded) operator to be C^0-scalar, with appropriate growth conditions (quasi-analytic or Stieltjes vectors) (Propositions 3.7 and 3.8, and Theorems 3.18~21). On a reflexive Banach lattice (Section 4), we get numerous necessary and sufficient conditions for an operator to be scalar, with positive projection-valued measure.

We have a good reason for focusing on Banach lattices, when discussing scalar, or C^0-scalar operators. In a future paper, we will show that when X is cyclic, with respect to a C^0-scalar operator, then there exists an ordering on X, and an equivalent norm, with respect to which X is a Banach lattice. With respect to this ordering, T generates a positive-definite group or a completely monotone semigroup (Deffinitions 3.5 and 3.6; also see Propositions 3.7 and 3.8 and Section 4). Thus, when T is scalar on X, X is "locally" a Banach lattice, and T "locally" has a positive projection-valued measure.

All operators are linear, on a Banach space (usually called "X"). $B(X)$ is the Banach algebra of all bounded linear operators on X.

2. Scalar Operators and Accretive Operators

We characterize unbounded scalar operators in terms of their resolvents existing and being scalar. Preliminaries on accretive operators may be found in [12], Section 10.8. Preliminaries on scalar operators may be found in [5] or [6].

Definition 2.1. A function, E, from the set of Borel measurable subsets of A, a Borel subset of the complex plane, into the set of projections on X, is a *projection-valued measure on* A if the following hold.

(1) $E(A)=I$.

(2) $E(B\cap C)=E(B)E(C)$, $\forall$ Borel sets B and C.

(3) $\exists M<\infty \ni \|E(B)\|<M$, $\forall$ sets B.

(4) $\forall x\in X$, E_x, defined by $E_x(B)\equiv E(B)x$, is a vector-valued measure, that is, if $<B_n>$ is a disjoint sequence of Borel sets, then $E_x(\bigcup B_n)=\Sigma_n E_x(B_n)$ with the series converging in norm.

Note that, for any $\phi\in X^*$, $x\in X$, $E_{x,\phi}$, defined by $E_{x,\phi}(B)\equiv\phi(E(B)x)$, is a complex-valued measure.

Definition 2.2. A bounded operator, T, is *scalar on* A if $\exists$ a projection-valued measure, E on A, $\ni$

$$Tx=\int_A t\; dE_x(t),\ \forall x\in X.$$

E is called the *projection-valued measure for* T. E is supported on the spectrum of T.

Definition 2.3. $D(T)\equiv$domain of the operator T. $C^\infty(T)\equiv\bigcap_{n=0}^{\infty}D(T^n)$. $\bar{T}$ is the closure of T. $D\subseteq D(T)$ is a *core* for T if $S\equiv T|_D$, T restricted to D, is $\ni \bar{S}=\bar{T}$.

Definition 2.4. A (possibly unbounded) operator, T, is *scalar on* $\boldsymbol{R}$ if $\exists$ a projection-valued measure, E, $\ni$

$$D(T)\equiv\{x\,|\lim_{n\to\infty}\int_{-n}^{n} t dE_x(t) \text{ exits}\},$$

with

$$T_x\equiv\lim_{n\to\infty}\int_{-n}^{n} t dE_x(t).$$

T is automatically closed.

We get a functional calculus for T: if f is a bounded, Borel measurable function on the spectrum of T, then $f(T)$, defined by

$$f(T)x \equiv \lim_{n\to\infty}\int_{-n}^{n} f(t)dE_x(t)$$

is a bounded operator. The map $f \mapsto f(T)$ is a continuous algebra homomorphism, that is, $fg(T)=f(T)g(T)$ and $\exists M<\infty \ni \|f(T)\| \leq M\|f\|_\infty$, $\forall f$, with the supremum taken over the spectrum of T.

T is *scalar on* $[0,\infty)$ if T is scalar on $\boldsymbol{R}$, and

$$Tx=\lim_{n\to\infty}\int_0^n tdE_x(t) \quad \forall x\in D(T).$$

Definition 2.5. T is *m.e.-accretive* if it generates a uniformly bounded semigroup, e^{-sT}, that is, $\exists M<\infty \ni \|e^{-sT}\| \leq M$, $\forall s\geq 0$.

T is m.e.-accretive if and only if T is m-accretive with respect to an equivalent norm (let $|||x||| \equiv \sup_{s\geq 0}\|e^{-sT}x\|$).

Theorem 2.6. T is scalar on $[0,\infty)$ if and only if T is m.e.-accretive, with $(1+T)^{-1}$ scalar on $[0,1]$.

Proof: Suppose T is scalar on $[0,\infty)$. Let E be the projection-valued measure for T.

$$(*) \qquad F(s)x \equiv \lim_{n\to\infty}\int_0^n e^{-st}dE_x(t)$$

defines a uniformly bounded semigroup $\{F(s)\}_{s\geq 0}$.

We will show that $F(s)$ is generated by T. It is convenient to work with a subspace of T.

Let $D\equiv\{E(A)x \mid x\in X,\ A$ is a bounded Borel subset of $[0,\infty)\}$.

Note that D is a subspace of x, since $E(A)x+E(B)y=E(A\cup B)(E(A)x+E(B)y)$.

We claim: (1) D is a core for T and (2) $\forall x\in D$, $-Tx=\frac{d}{ds}F(s)x|_{s=0}$.

To show (1), suppose $x\in D(T)$. For any n, let $x_n \equiv E([0,n])x$. x_n is in D, and $Tx_n=\int_0^n tdE_x(t)$, thus $x_n\to x$ and $Tx_n\to Tx$, as desired.

To show (2), suppose A is a bounded Borel subset of $[0,\infty)$. For any x, $T(E(A)x) =\int_A tdE_x(t)$, $F(s)E(A)x=\int_A e^{-st}dE_x(t)$. As a function of t, $\frac{1}{s}(e^{-st}-1)$ converges uniformly to $(-t)$, on A as $s\to 0$, because A is bounded. This implies that

$$\lim_{s\to 0}\frac{1}{s}(F(s)E(A)x-E(A)x)$$
$$=\int_A \lim_{s\to 0}\frac{1}{s}(e^{-st}-1)dE_x(t)=\int_A -tdE_x(t)$$
$$=-T(E(A)x) \text{ as claimed.}$$

Since T is closed, (1) and (2) above imply that T is m.e.-accretive, and $e^{-sT}=F(s)$ (see*).

Let $g(t)\equiv(1+t)^{-1}$. Define a projection-valued measure, M, on $[0,1]$, by

$$M(A)\equiv E(g^{-1}(A)).$$

For any x, $(1+T)^{-1}x=\int_0^\infty e^{-s}e^{-sT}xds$

$$=\int_0^\infty e^{-s}\int_0^\infty e^{-st}dE_x(t)ds$$
$$=\int_0^\infty\int_0^\infty e^{-s(1+t)}ds\ dE_x(t)=\int_0^\infty g(t)dE_x(t)$$
$$=\int_0^1 wdM_x(w).$$

Thus $(1+T)^{-1}$ is scalar on $[0,1]$.

Conversely, suppose T is m.e.-accretive, with $(1+T)^{-1}$ scalar on $[0,1]$.

Let F be the projection-valued measure for $(1+T)^{-1}$.

Define a projection-valued measure, E, by $E(A)\equiv F(g(A))$ $(g(t)\equiv(1+t)^{-1})$.

Let $Rx\equiv\lim_{n\to\infty}\int_0^n tdE_x(t)$, $\forall x\in DR)\equiv\{x|\lim_{n\to\infty}\int_0^n t\ dD_x(t)$ exits$\}$.

By the first half of this proof, R is m.e.-accretive, with $(1+R)^{-1}=g(R)=(1+T)^{-1}$.

Since R and T are m.e.-accretive, and $(1+R)^{-1}=(1+T)^{-1}$, $R=T$, that is, T is scalar on $[0,\infty)$. ///

Essentially the same proof, with the Fourier transform replacing the Laplace transform, gives the following.

Theorem 2.7. T is scalar on the real line if and only if (iT) is m.e.-accretive, with $(1-iT)(1+iT)^{-1}$ scalar on the unit circle in the complex plane.

3. C^0-Scalar Operators on Banach Lattices

A bounded operator is C^0-scalar if it has a functional calculus defined for all functions continuous on its spectrum. With the theorems of the previous section in mind, we present definitions for (possibly unbounded) operators to be C^0-scalar on $[0,\infty)$, or C^0-scalar on the real line (Definition 3.2).

In this section, we assume that X is a Banach lattice, with positive cone X^+. Basic material on Banach lattice may be found in [15].

In Definitions 3.5 and 3.6, we define positive-definite and completely monotone operator-valued functions. We show that a generator of a positive-definite group is C^0-scalar on the real line and a generator of a completely monotone semigroup is C^0-scalar on $[0,\infty)$ (Propositions 3.7 and 3.8). On a reflexive space, when C^0-scalar is equivalent to scalar, being such a generator is equivalent to having a positive projection-valued measure (see Section 4). Using the classical analysis of quasi-analytic functions and the moment problem (a summary is given in Definitions 3.12 through 3.17), we present sufficient conditions for an operator to generate a positive-definite group-the sequence $<\phi(T^n x)>_{n=0}^{\infty}$ must be positive-definite for sufficiently many positive ϕ in X^*, x in X. When the sequence equals the moments of a positive measure supported on $[0,\infty]$, then T generates a completely monotone semigroup (see Theorems 3.18~3.21). Converses appear in Section 4. These theorems make no assumptions about the operator being a generator or having any resolvents.

Definition 3.1. If D is a bounded subset of the complex plane, then a bounded operator, T, is C^0*-scalar on* D if there exists a continuous algebra homomorphism, Λ, from $C(D)$ into $B(X)$, such that $\Lambda(f_0)=I$, $\Lambda(f_1)=T$, where $f_0(z)\equiv 1$, $f_1(z)\equiv z$.

An operator scalar on D is C^0-scalar on D. When X is reflexive, the converse is also true (see [5]).

There are obvious problems in extending this definition directly to unbounded operators. Motivated by Theorems 2.6 and 2.7, we make the following definitions.

Definition 3.2. A (possibly unbounded) operator, T, is C^0*-scalar on* $[0,\infty)$ if T is m.e.-accretive (Definition 2.5) and $(1+T)^{-1}$ is C^0-scalar on $[0,1]$.

T is C^0*-scalar on* $\boldsymbol{R}$ if iT is m.e.-accretive and $(1-iT)(1+iT)^{-1}$ is C^0-scalar on the unit circle.

Definition 3.3. D is a *cone* if $\alpha x+By\in D$, whenever $x,y\in D$, $\alpha,B\geq 0$. $F:D\to C$ we will call *linear* if $F(\alpha x+By)=\alpha F(x)+BF(Y)$, for $\alpha,B\geq 0$, $x,y\in D$; we will use the analogous definition for bilinear.

The following technical lemma is an elementary application of facts about Banach lattices.

Lemma 3.4. Suppose D is a cone dense in X^+.

(a) Suppose $F: Dx(X^*)^+\to C$ is bilinear, and $|F(x,\phi)|\leq\|\phi\|\,\|x\|$, $\forall(x,\phi)\in Dx(X^*)^+$. Then $\exists!$ bounded $T: X\to X^{**}\ni(Tx)(\phi)=F(x,\phi)$, $\forall(x,\phi)\in Dx(X^*)^+$. $\|T\|\leq 16$.

(b) If $T\in B(X)$ is $\ni|\phi(Tx)|\leq\|\phi\|\,\|x\|$, $\forall(x,\phi)\in Dx(X^*)^+$, then $\|T\|\leq 16$.

Definition 3.5. F: $R\to B(X)$ is *positive-definite* if, for any finite sequence $<\alpha_k>$ of complex numbers, the operator

$$\sum_{k,j}\alpha_k\bar{\alpha}_j F(s_k-s_j)$$

is positive. This is equivalent to saying that, for any positive ϕ,x, the complex-valued function $s\to\phi(F(s)x)$ is positive-definite.

Definition 3.6. G: $(0,\infty)\to B(X)$ is *completely monotone* if, for any positive ϕ,x, the function $s\to\phi(G(s)x)$ is completely monotone, that is, for any $n,s>0$, $\frac{d^n}{ds^n}\phi(G(s)x)$ exists, and

$$(-1)^n\frac{d^n}{ds^n}\phi(G(s)x))\geq 0.$$

Proposition 3.7. If T generates a positive-definite group e^{-isT}, then T is C^0-scalar on $\boldsymbol{R}$.

Proof: For any positive ϕ,x, by Bochner's theorem, there exists a unique positive finite measure $E_{x,\phi}$, so that, for s real.

$$(*)\qquad \phi(e^{-isT}x)=\int_R e^{-ist}dE_{x,\phi}(t).$$

By Lemma 3.4 and elementary estimates, iT and $(-iT)$ are m.e.-accretive.

Let $g(t)\equiv(1-it)(1+it)^{-1}$, $S\equiv g(T)\equiv(1-iT)(1+iT)^{-1}$. I claim that, for any integer k, $\phi\in(X^*)^+$, $x\in C^\infty(T)\cap X^+$,

$$(**)\qquad \phi(S^k x)=\int_R[g(t)]^k dE_{x,\phi}(t).$$

The calculation follows.

$$\phi(S^k x)=\phi(I-2iT(1+iT)^{-1})_x^k$$
$$=\sum_{m=0}^{k}\binom{k}{m}(-2i)^m\phi(T^m(1+iT)_x^{-m})$$
$$=\sum_{m=0}^{k}\binom{k}{m}\frac{(-2i)^m}{(m-1)!}\int_0^\infty s^{m-1}e^{-s}\phi(T^m e^{-isT}x)ds$$
$$=\sum_{m=0}^{k}\binom{k}{m}\frac{(-2i)^m}{(m-1)!}\int_0^\infty s^{m-1}e^{-s}\int\left(\frac{-id}{ds}\right)^m\phi(e^{-isT}x)ds$$
$$=\sum_{m=0}^{k}\binom{k}{m}\frac{(-2i)^m}{(m-1)!}\int_0^\infty s^{m-1}e^{-s}\int_R t^m e^{-ist}dE_{x,\phi}(t)ds \quad \text{(by *)}$$
$$=\sum_{m=0}^{k}\binom{k}{m}\int_R\frac{(-2it)^m}{(m-1)!}\int_0^\infty s^{m-1}e^{-s(1+it)}ds\ dE_{x,\phi}(t)$$
$$=\int_R\sum_{m=0}^{k}\binom{k}{m}\frac{(-2it)^m}{(1+it)^m}dE_{x,\phi}(t)$$
$$=\int_R[g(t)]^k dE_{x,\phi}(t),\ \text{as claimed.}$$

Let T be the unit circle, in the complex plane, and let p be any polynomial, in z and $\frac{1}{z}$, on T.

By (**), we have

$$(***)\qquad \phi(p(S)x)=\int_T p(w)dE_{x,\phi}(g^{-1}(w)),$$

for any $\phi\in(X^*)^+$, $x\in C^\infty(T)\cap X^+$.

To apply Lemma 3.4, we need to show that $C^\infty(T)\cap X^+$ is dense in X^+.

To do this, define, for any $x, \Lambda_n x$, by

$$\Lambda_n x\equiv\sqrt{\frac{n}{\Pi}}\int_R e^{-ns^2}e^{-isT}xds.$$

Since

$$e^{-irt}\Lambda_n x=\sqrt{\frac{n}{\Pi}}\int_R e^{-ns^2}e^{-i(s+r)T}xds$$
$$=\sqrt{\frac{n}{\Pi}}\int_R e^{-n(w-r)^2}e^{-iwT}xdw,\ \text{a } C^\infty\text{-function of } r,\ \Lambda_n x\in C^\infty(T),$$

for any n, x.

For ϕ, x positive, we have

$$\phi(A_n x) = \sqrt{\frac{n}{\Pi}} \int_R e^{-ns^2} \int_R e^{-ist} dE_{x,\phi}(t) ds$$
$$= \int_R \sqrt{\frac{n}{\Pi}} \int_R e^{-ns^2} e^{-ist} ds\, dE_{x,\phi}(t)$$
$$= \int_R e^{-t^2/n} dE_{x,\phi}(t).$$

Thus $A_n x$ is positive whenever x is. This calculation also shows that, by monotone convergence, $\lim_{n\to\infty} \phi(A_n x) = \phi(x)$, for all positive ϕ, x.

Thus $\{A_n x \mid x \in X^+,\ n \in N\}$ is weakly dense in X^+; since X^+ is convex, the set is also strongly dense.

Thus $\{A_n x \mid x \in X^+,\ n \in N\}$ is contained in $C^\infty(T) \cap X^+$, and is dense in X^+.

By (***), we have, for $\phi \in (X^*)^+$, $x \in C^\infty(T) \cap X^+$, p any polynomial in z and $\frac{1}{z}$, $|\phi(p(S))x| \leq \|p\|_\infty E_{x,\phi}(\boldsymbol{R}) = \|p\|_\infty \phi(x) \leq \|p\|_\infty \|\phi\|\ \|x\|$.

By Lemma 3.4, $\|p(S)\| \leq 16 \|p\|_\infty$.

Since the set of all polynomials in z and $\frac{1}{z}$ is dense in $C(T)$, the map $p \to p(S)$ extends uniquely from the polynomials to a continuous algebra homomorphism from $C(T)$ into $B(X)$.

Thus S is C^0-scalar on T, so that T is C^0-scalar on $\boldsymbol{R}$. ///

Proposition 3.8. If T generates a completely monotone semigroup e^{-sT}, ther T is C^0-scalar on $[0, \infty)$.

Proof: Let $g(t) \equiv (1+t)^{-1}$, $S \equiv g(T) \equiv (1+T)^{-1}$. If ϕ, x are positive, then by Bernstein's theorem, $\exists !$ positive measure $E_{x,\phi} \ni$

(*) $$\phi(e^{-sT}x) = \int_0^\infty e^{-st} dE_{x,\phi}(t), \quad \forall s > 0.$$

Note that, by monotone convergence, $E_{x,\phi}[0, \infty) = \lim_{s\to 0} \int_0^\infty e^{-st} dE_{x,\phi}(t)$ $= \phi\left[\lim_{s\to 0}(e^{-sT}x)\right] = \phi(x)$.

Thus $E_{x,\phi}$ is finite, for all positive x, ϕ.

As in the proof of the previous proposition, we calculate that

$$\phi(S^n x) = \int_0^\infty [g(t)]^n dE_{x,\phi}(t),$$

for all nonnegative integers n, so that

$$\phi(p(S)x) = \int_0^1 p(w) dE_{x,\phi}(g^{-1}(w)),$$

for any polynomial on $[0,1]$, x, ϕ positive.

As in the previous proposition, it follows that S is C^0-scalar on $[0,1]$, so that T is C^0-scalar on $[0,\infty)$. ///

Definition 3.9. Suppose T is an operator on X and S is an operator on Y. We define S to be an *extension* of T if $\exists$ an embedding, Λ, of X in Y (Λ: $X \to Y$ is one-to-one and bicontinuous) $\ni \Lambda(D(T)) \subseteq D(S)$, and $\Lambda(Tx) = S(\Lambda x)$, $\forall x \in D(T)$.

Lemma 3.10. Suppose T is a densely defined operator on X and D is a core for T.

(a) If $\exists$ a uniformly bounded family $\{F(s) \mid s \text{ is real}\}$ of operators from $X \to X^{**} \ni$, $\forall x \in D$, $F(s)x$ is differentiable, with

$$\frac{d}{ds} F(s)x = iF(s)Tx, \text{ and } F(0) = I,$$

then $\bar{T}$ has an extension, $\tilde{T}$, $\ni \pm i\tilde{T}$ is m-accretive.

(b) If $\exists$ a uniformly bounded family $\{G(z) \mid Re(z) \geq 0\}$ of operators from $X \to X^{**} \ni$ $\forall x \in D$, $G(z)x$ is analytic with

$$\frac{d}{dz} G(z)x = -G(z)Tx, \quad G(0) = I,$$

then $\bar{T}$ has an extension that generates a uniformly bounded holomorphic semigroup.

Proof: (a) Let $Z \equiv \{\text{bounded continuous } f\colon \boldsymbol{R} \to X^{**}\}$, $Y \equiv \{\text{uniformly continuous } f \in Z\}$, with $\|f\| \equiv \sup_{s \in R} \|f(s)\|$.

Define Λ: $X \to Z$ by

$$(\Lambda x)(s) \equiv F(s)x.$$

This embeds X in Z; we claim that Λ embeds X in Y.

If $x \in D$, the Λx is differentiable, with $\|\frac{d}{ds} \Lambda x\| \leq [\sup_{s \in R} \|F(s)\|] \|Tx\|$.

Since Λx has a uniformly bounded derivative, $(\Lambda x) \in Y$.

Thus $\Lambda(D) \subseteq Y$; since Λ is continuous, and D is dense in X, $\Lambda(X) \subseteq Y$, proving the claim.

Note that $\Lambda(Tx) = i\frac{d}{ds}\Lambda x$. Letting $\tilde{T} \equiv i\frac{d}{ds}$, on Y, we have that $\pm i\tilde{T}$ is m-accretive.

(b) Let $H \equiv \{x \in C \mid Re(z) \geq 0\}$, $Y \equiv \{\text{bounded uniformly continuous analytic } f\colon H \to X^{**}\}$.

This proof is identical to the proof of (a).

Example 3.11. Let $X \equiv C_0[0,1]$, $A \equiv -i\frac{d}{dx}$, with $D(A) \equiv \{\text{differentiable } f \in C_0[0,1] \mid f' \in C_0[0,1]\}$.

Let $F(s) \equiv$ translation group, $(F(s)f)(x) \equiv f(s+x)$.

A and $\{F(s)\}$ satisfy the hypothesis of Lemma 3.10, but $i\bar{A}$ is not m-accretive. ///

In the next six definitions, we list some results from classical analysis, along with some recent applications to operator theory.

Definition 3.12. A sequene $\langle M_n \rangle_{n=0}^{\infty}$ of positive numbers is *quasi-analytic* if:

$$[\|f^{(k)}\|_\infty \leq M_k,\ f^{(k)}(0)=0,\ \forall k] \text{ implies that } f(t)=0,\ \forall \text{ real } t.$$

$$\langle M_n \rangle_{n=0}^{\infty} \text{ is quasi-analytic if and only if } \sum_{n=0}^{\infty} [\inf_{k \leq n} M_k^{-1/k}] = \infty.$$

(Denjoy-Carleman theorem, [13], p. 412).

A vector $x \in C^\infty(T)$ is a *quasi-analytic vector* if $\langle \|T^n x\| \rangle_{n=0}^{\infty}$ is quasi-analytic.

$$Dq(T) \equiv \{\text{quasi-analytic vectors for } T\}.$$

If T has an m.e.-accretive extension, and $Dq(T)$ is dense in X, then $\bar{T}$ is m.e.-accretive (see [4]). The same argument used in [4] shows that it's sufficient to have $Dq(T) \cap X^+$ dense in X^+.

Definition 3.13. A sequence $\langle M_n \rangle_{n=0}^{\infty}$ of positive numbers satisfies a *Stieltjes growth condition* if

$$\sum_{n=0}^{\infty} [\inf_{k \leq n} M_k^{-1/2k}] = \infty.$$

Let $U \equiv \{z \in C \mid Im(z) > 0\}$. If $[f^{(k)} \in H^\infty(U),\ \|f^{(k)}\|_\infty \leq M_k,\ f^{(k)}(0)=0,\ \forall k]$ and $\langle M_k \rangle$ satisfies a Stieltjes growth condition, then $f(z)=0,\ \forall z \in U$ (see [9]). A vector $x \in C^\infty(T)$ is a *Stieltjes vector* if $\langle \|T^n x\| \rangle_{n=0}^{\infty}$ satisfies a Stieltjes growth condition.

$$D_s(T) \equiv \{\text{Stieltjes vectors for } T\}.$$

If T has an extension that generates a bounded holomorphic semigroup and $D_s(T) \cap X^+$ is dense in X^+, then $\bar{T}$ generates a bounded holomorphic semigroup (see [4]).

Definition 3.14. A vector $x \in C^\infty(T)$ is an *analytic vector* if $\exists M, K$ so that

$$\|T^n x\| \leq KM^n(n!), \ \forall n.$$

A vector $x \in C^\infty(T)$ is a *semi-analytic vector* if

$$\|T^n x\| \leq KM^n(2n)!, \ \forall n$$

$$D_a(T) \equiv \{\text{analytic vectors for } T\}$$

$$D_{sa}(T) \equiv \{\text{semi-analytic vectors for } T\}.$$

We have the following relationships between these growth conditions.

$$\begin{array}{ccc} D_a(T) & \subseteq & D_q(T) \\ \cap\!\mid & & \mid\!\cap \\ D_{sa}(T) & \subseteq & D_s(T). \end{array}$$

Definition 3.15. A sequence $\langle a_n \rangle_{n=0}^\infty$ of real numbers is *positive-definite* if

$$\sum_{k,j} \alpha_k \bar{\alpha}_j a_{k+j} \geq 0,$$

for all finite sequences $\langle \alpha_k \rangle$ of complex numbers.

Definition 3.16. If u is a measure on the real line, then the set of *moments* of u, is a sequence $\langle a_n \rangle_{n=0}^\infty \ni$

$$a_n = \int_R t^n du(t) \ \forall n.$$

The sequence $\langle a_n \rangle_{n=0}^\infty$ is positive-definite if and only if $\langle a_n \rangle$ equals the set of moments of a positive Borel measure on the real line.

If, in addition, $\langle |a_n| \rangle_{n=0}^\infty$ is quasi-analytic, then the positive measure in *unique.* (See [16], and Defintion 3.12.)

Definition 3.17. A sequence $\langle a_n \rangle_{n=0}^\infty$ of real numbers satisfies a *Stieltjes moment condition* if and only if

$$\sum_{k,j} \alpha_k \bar{\alpha}_j a_{k+j} \geq 0, \ \sum_{k,j} \alpha_k \bar{\alpha}_j a_{k+j+1} \geq 0,$$

for all finite sequences $\langle \alpha_k \rangle$ of complex numbers.

A sequence $\langle a_n\rangle_{n=0}^{\infty}$ satisfies a Stieltjes moment condition if and only if $\langle a_n\rangle$ equals the set of moments of a positive Borel measure supported on $[0,\infty)$.

If, in addition, $\langle a_n\rangle_{n=0}^{\infty}$ satisfies a Stieltjes growth condition, then the positive measure is unique (see [16] and Definition 3.13).

The following theorems present sufficient conditions for an operator to generate a positive-definite group, or a completely monotone semigroup (see Proposition 3.7 and 3.8).

Theorem 3.18. Suppose $D\subseteq X^+\cap D_s(T)$ has the following properties:

(1) D is dense in X^+.

(2) The sequence $\langle\phi(T^n x)\rangle_{n=0}^{\infty}$ satisfies a Stieltjes moment condition, $\forall\phi\in(X^*)^+$, $x\in D$.

(3) Span (D) is a core for T.

(4) D is convex.

Then $\bar{T}$ generates a completely monotone semigroup.

Proof: By (2), for any $(x,\phi)\in Dx(X^*)^+$, $\exists!$ (since $x\in Ds(T)$) positive measure $E_{x,\phi}\ni$ (see Definition 3.17).

$$(*)\qquad \phi(T^n x)=\int_0^{\infty} t^n\, dE_{x,\phi}(t),\quad \forall n\in N.$$

When $x\in D_s(T)$ and $\alpha>0$, then αx and Tx are in $D_s(T)$. Thus, by hypothesis (4), we may assume that D is a cone, and $T(D)\subseteq D$.

Note that, by uniqueness of the positive measure in (*), $E_{Tx,\phi}(t)=tE_{x,\phi}(t)$, $E_{x,\phi}+E_{x,\psi}=E_{x,\phi+\psi}$, and $E_{x,\phi}+E_{y,\phi}=E_{x+y,\phi}$, for $\phi,\psi\in(X^*)^+$, $x,y\in D$.

This means that, if $Re(z)>0$, $G(z)$, from X to X^{**}, may be defined, using Lemma 3.4, by

$$(**)\qquad (G(z)x)(\phi)\equiv\int_0^{\infty} e^{-zt}dE_{x,\phi}(t)$$

for any $(x,\phi)\in Dx(X^*)^+$.

Since $|(G(z)x)(\phi)|\le E_{x,\phi}([0,\infty))=\phi(x)\le\|\phi\|\,\|x\|$, $\{G(z)\,|\,Re(z)>0\}$ is a uniformly bounded family of operators.

To apply Lemma 3.10(b), we must verify that the map $z\mapsto G(z)x$ is analytic, for all $x\in D$, and that

$$(***)\qquad \frac{d}{dz}G(z)x=-G(z)Tx.$$

The calculation follows.

$$\begin{aligned}
&\left|\left(\frac{1}{h}(G(z+h)x-G(z)x)+G(z)Tx\right)(\phi)\right| \\
&=\left|\int_0^\infty \frac{1}{h}(e^{-(z+h)t}-e^{-zt})dE_{x,\phi}(t)+\int_0^\infty e^{-zt}dE_{Tx,\phi}(t)\right| \\
&=\left|\int_0^\infty\left[\frac{e^{-ht}-1}{h}-t\right]e^{-zt}\,dE_{x,\phi}(t)\right| \\
&\leq\int_0^\infty ht^2dE_{x,\phi}(t),\ \text{by the mean-value theorem, applied twice,} \\
&=h\phi(T^2x)\leq h\|\phi\|\ \|T^2x\|.
\end{aligned}$$

Thus $\|\frac{1}{h}(G(z+h)x-G(z)x)+G(z)Tx\|\leq h\|T^2x\|$, $\forall h>0$, proving (***).

By Lemma 3.10(b), T has an extension that generates a uniformly bounded holomorphic semigroup. By hypothesis (1) of this theorem, $\bar{T}$ generates a uniformly bounded holomorphic semigroup. (See Definition 3.13.) By (***), $G(z)=e^{-zT}$.

To see that $G(z)$ is completely monotone, fix $k\in N$. If $x\in D$, then, for $\phi\in(X^*)^+$, $(-1)^k\frac{d^k}{ds^k}\phi(G(s)x)=\int_0^\infty t^ke^{-st}dE_{x,\phi}(t)\geq 0$, by (**). Since $G(z)$ is holomorphic, $T^kG(s)$ is a bounded operator, $\forall s>0$, and $(-1)^k\frac{d^k}{ds^k}G(s)x=T^kG(s)x$, $\forall x\in X$. Since D is dense in X^+, and $T^kG(s)x$ is positive, $\forall x\in D$, $T^kG(s)x$ is positive $\forall x\in X^+$, so that $(-1)^k\frac{d^k}{ds^k}\phi(G(s)x)\geq 0$, for all positive ϕ, x as desired. This concludes the proof of the theorem. ///

Since $D_{sa}(T)$ (see Definition 3.14.), unlike $D_s(T)$, is always a vector space, we may remove convexity as a hypothesis in the following.

Theorem 3.19. Suppose $D\subseteq X^+\cap D_{sa}(T)$ has the following properties.

(1) D is dense in X^+.

(2) $\langle\phi(T^nx)\rangle_{n=0}^\infty$ satisfies a Stieltjes moment condition, $\forall(x,\phi)\in Dx(X^*)^+$.

(3) span(D) is a core for T.

Then $\bar{T}$ generates a completely monotone semigroup.

Theorem 3.20. Suppose $D\subseteq X^+\cap D_q(T)$ has the following properties.

(1) D is dense in X^+.

(2) The sequence $\langle\phi(T^nx)\rangle_{n=0}^\infty$ is positive-definite, $\forall\phi\in(X^*)^+$, $x\in D$.

(3) span(D) is a core for T.

(4) D is convex.

Then $\bar{T}$ generates a positive-definite group $e^{-is\bar{T}}$.

Proof: The major difference between this proof and the proof of Theorem 3.18. is that $T(D)$ is not contained in D. Thus, to apply Lemma 3.10(a), more care is required in constructing the family of operators $\{F(s) \mid s \text{ is real}\}$.

Let $p^+ \equiv \{\text{real-valued polynomials, } p \mid p(t) \geq 0 \ \forall \text{ real } t\}$.

Using Zorn's lemma, choose $C \subseteq D \ni (x, \ y \in C, \ x \neq y)$ implies that $\{p(T)x \mid p \in P^+\}$ is disjoint from $\{p(T)y \mid p \in P^*\}$, and $D \subseteq \{p(T)x \mid p \in P^+, \ x \in C\}$.

If $(x, \phi) \in Cx(X^*)^+$, then by hypothesis (2), $\exists!$ (since $x \in Dq(T)$) positive measure $E_{x,\phi} \ni$ (see Definition 3.16.)

$$\phi(T^n x) = \int_R t^n dE_{x,\phi}(t) \ \forall n \in N.$$

As in the proof of Theorem 3.18, for any real s, $F(s)$, from X to X^{**}, may be defined by

$$((*) \qquad (F(s))\, p(T)x)(\phi) \equiv \int_R e^{-ist} p(t)\, dE_{x,\phi}(t)$$

for $(x, \phi) \in Cx(X^*)^+$, p any polynomial.

As in Theorem 3.18, $\{F(s) \mid s \text{ is real}\}$ is uniformly bounded, and $\forall x \in D$, the function $s \mapsto F(s)x$ is differentiable, with

$$(**) \qquad \frac{d}{ds} F(s)x = -iF(s)Tx.$$

By Lemma 3.10(a), T has an extension $\tilde{T} \in \pm i\bar{T}$ is m-accretive. By hypothesis (1) of this theorem, $\pm i\bar{T}$ is m.e.-accretive. (See Definition 3.12.) By $(**)$, $e^{-is\bar{T}} = F(s)$.

To verify that $F(s)$ is positive-definite, let $\langle \alpha_k \rangle$ be a finite sequence of complex numbers. By $(*)$, $\sum_{k,j} \alpha_k \bar{\alpha}_j F(s_k - s_j)x$ is positive, $\forall x \in D$. Since D is dense in X^+, it follows that $\sum \alpha_k \bar{\alpha}_j F(s_k - s_j)$ is a positive operator, as desired. ///

Since $D_a(T)$ (see Definition 3.14.) is a vector space, we get the following.

Theorem 3.21. Suppose $D \subseteq X^+ \cap D_a(T)$ has the following properties.

(1) D is dense in X^+.

(2) The sequence $\langle \phi(T^n x) \rangle_{n=0}^{\infty}$ is positive-definite, $\forall \phi \in (X^*)^+$, $x \in D$.

(3) span(D) is a core for T.

Then $\bar{T}$ generates a positive-definite semigroup $e^{-it\bar{T}}$.

4. Scalar Operators on Reflexive Banach Lattices

In this section, we assume that X is a reflexive Banach lattice, and derive numerous equivalent characterizations of operators scalar on $[0,\infty)$, or the real line, with positive projection-valued measure.

Corollary 4.5. is a generalization of Bernstein's theorem and Bochner's theorem to operator-valued functions.

Definition 4.1. T is *m-dispersive* if T is m-accretive, and e^{-sT} is a positive operator, for all $s \geq 0$.

We need an appropriate definition for an unbounded operator being positive.

Definition 4.2. T is *positive* if $D(T) \cap X^+$ is dense in X^+, and $\forall x \in D(T) \cap X^+$, $(Tx) \in X^+$.

Definition 4.3. Suppose X is a reflexive Banach lattice. The following are equivalent.

(1) T generates a completely monotone semigroup.

(2) T is scalar on $[0,\infty)$, with positive projection-valued measure.

(3) T is closed, and $\exists D \subseteq D_a(T) \cap X^+ \ni \text{span}(D)$ is a core for T, $\bar{D} = X^+$, and $\langle \phi (T^n x) \rangle_{n=0}^{\infty}$ satisfies a Stieltjes moment condition $\forall \phi \in (X^*)^+$, $x \in D$.

(4) T is positive and m-dispersive and generates a bounded holomorphic semigroup.

Proof: (1)→(2). In Proposition 3.8, we showed that T is C^0-scalar on $[0,\infty)$. Since X is reflexive, T is scalar on $[0,\infty)$. Let E be the projection-valued measure for T.

As in (*) of the proof of Theorem 2.6, e^{-sT} is given by

$$(*) \qquad e^{-sT}x = \int_0^\infty e^{-st} dE_x(t).$$

For positive ϕ, x, we thus have

$$\phi(e^{-sT}x) = \int_0^\infty e^{-st} dE_{x,\phi}(t);$$

since $s \to \phi(e^{-sT}x)$ is completely monotone, $E_{x,\phi}$ is a positive measure. Thus E is a positive projection-valued measure.

(2)→(1). Let E be the projection-valued measure for T. $(-1)^n \frac{d^n}{ds^n} \phi(e^{-sT}x) = \int_0^\infty t^n e^{-st} dE_{x,\phi}(t)$ (see (*) in (1)→(2)), which is nonnegative, $\forall n$, whenever x and ϕ are positive, because E is positive. Thus e^{-sT} is completely monotone.

(3)→(1) is **Theorem** 3.19 (**see Definition** 3.14).

(1)→(4). (1) obviously implies that T is m-dispersive. (1) implies (2), which implies that T is positive and generates a bounded holomorphic semigroup $e^{-zT}x \equiv \int_0^\infty e^{-zt} dE_x(t)$, whee $Re(z)>0$.

(4)→(1). Suppose $x \in X^+$. Since T generates a bounded holo morphic semigroup, $e^{-sT}x \in C^\infty(T)$, $\forall s>0$. Since T is m-dispersive, $e^{-sT}x \in X^+$. Thus, for any n, $(-1)^n \frac{d^n}{ds^n}(e^{-sT}x) = T^n(e^{-sT}x)$ is positive.

(2)→(3). Let E be the projection-valued measure for T. Let $D \equiv \{E(A)x \mid x \in X^+$, A is a bounded Borel subset of $[0,\infty)\}$.

As in the proof of Theorem 2.6, D is dense in X^+, and span(D) is a core for T.

To see that $D \subseteq D_a(T)$, suppose A is a bounded Borel subset of $[0,\infty)$. Let $K \equiv \sup\{x \in A\}$.

$\|T^n(E(A)x)\| = \|\int_A t^n dE_x(t)\| \leq K^n \|E(A)x\|$, for any $x \in X^+$, so that $E(A)x \in D_a(T)$.

For any positive ϕ, since $E(A)dE_{\phi,x}(t)$ is a positive measure, for any $x \in X^+$, $\langle \phi\,(T^n x)\rangle_{n=0}^\infty$ satisfies a Stieltjes moment condition, $\forall x \in D$. ///

Analogous proofs, with the Fourier transform replacing the Laplace transfrom, give the following.

Theorem 4.4. Suppose X is a reflexive Banach lattice. The following are equivalent.

(1) T generates a positive-definite group, e^{-isT}.

(2) T is scalar on R, with positive projection-valued measure.

(3) T is closed, and $\exists D \subseteq D_a(T) \cap X^+ \ni \bar{D} = X^+$, span$(D)$ is a core for T, and $\langle \phi\,(T^n x)\rangle_{n=0}^\infty$ is positive-definite, $\forall \phi \in (X^*)^+$, $x \in D$.

The statements and proofs of (1)↔(2), in the preceding theorems, gives the following.

Corollary 4.5. Suppose X is a reflexive Banach lattice.

(a) F: $\mathbf{R} \to B(X)$ is a positive-definite group if and only if F is the Fourier transform of a positive projection-valued measure.

(b) G: $[0,\infty) \to B(X)$ is a completely monotone semigroup if and only if G is the Laplace transform of a positive projection-valued measure.

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