Comm. Korean Math. Soc. 1 (1986)

EVALUATION SUBGROUPS OF THE HOMOTOPY GROUPS OF A TRANSFORMATION GROUP

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1. Introduction

In [3], [4], F. Rhodes introduced the homotopy groups of a transformation group (X, G) as a generalization of the homotopy groups of a topological space X. D.H. Gottlieb ([1], [2]) introduced the evaluation subgroups of the homotopy groups of a topological space. In [5], the author introduced the subgroup $E_n(X, *, G)$ and examined the properties of $E_1(X, *, G)$ mainly.

In this paper, we expand this subgroups to the evaluation subgroups $E_n(X, A, *, G)$ of $\sigma_n(X, *, G)$ which contain $E_n(X, *, G)$ and $\sigma_n(X, *, G)$ as special cases. The relation between $E_n(X, A, *, G)$ and the evaluation map from the mapping space X^A to X is examined and some properties of $E_n(X, A, *, G)$ are shown.

2. Notations

The n-dimensional cube will be denoted by $I^n=\{(t_1, t_2, \cdots, t_n) \mid 0 \leqslant t_i \leqslant 1$ for $1 \leqslant i \leqslant n\}$. By identifying corresponding points of pairs of opposite faces of the cube we obtain as a quotient space the n-dimensional torus T^n. It will be convenient to use the notation $\hat{t}_i = t_i \pmod 1$, $T^n = \{(\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_n)\}$. The cylinders $C^n = \{(t_1, \hat{t}_2, \cdots, \hat{t}_n)\}$ obtained by identifying corresponding points all but the first pair of opposite faces of the cube will be used in the definition of the homotopy groups of a transformation group. Let X be a topological space with base point $*$. If $r<n$, a map $f: C^r \to X$ gives rise to a map $f^n: C^n \to X$ defined by $f^n(t_1, \hat{t}_2, \cdots, \hat{t}_r, \cdots, \hat{t}_n) = f(t_1, \hat{t}_2, \cdots, \hat{t}_r)$. If $\rho(t)=1-t$, $\rho^n(t_1, \hat{t}_2, \cdots, \hat{t}_n) = (1-t_1, \hat{t}_2, \cdots, \hat{t}_n)$.

Given a transformation group $(X, *, G)$, for each element g of G, let

This research is supported by Korea Science and Engineering Foundation Grant.

the set $C^n(X, *, g)$ consists of all the continuous maps $f: C^n \to X$ such that

$$f(0, \hat{t}_2, \cdots, \hat{t}_n) = *$$
$$f(1, \hat{t}_2, \cdots, \hat{t}_n) = g(*).$$

Two sets $C^n(X, *, g_1)$ and $C^n(X, *, g_2)$ are formally distinct even if $g_1(*) = g_2(*)$. The elements of the set will be called maps of C^n of order g.

Two maps f_0 and f_1 of C^n of order g are said to be *homotopic* if there exists a continuous map $F: C^n \times I \to X$ such that

$$F(t_1, \hat{t}_2, \cdots, \hat{t}_n, 0) = f_0(t_1, \hat{t}_2, \cdots, \hat{t}_n),$$
$$F(t_1, \hat{t}_2, \cdots, \hat{t}_n, 1) = f_1(t_1, \hat{t}_2, \cdots, \hat{t}_n)$$

for $(t_1, \hat{t}_2, \cdots, \hat{t}_n) \in C^n$,

$$F(0, \hat{t}_2, \cdots, \hat{t}_n, t) = *,$$
$$F(1, \hat{t}_2, \cdots, \hat{t}_n, t) = g(*)$$

for $0 \leqslant t \leqslant 1$, $0 \leqslant t_i \leqslant 1$ and $2 \leqslant i \leqslant n$.

The homotopy class of a map f of order g will be denoted by $[f;g]$. If $f_1, f_2 : C^n \to X$ are maps such that

$$f_1(1, \hat{t}_2, \cdots, \hat{t}_n) = f_2(0, \hat{t}_2, \cdots, \hat{t}_n),$$

then the sum of the two maps, $(f_1 + f_2) : C^n \to X$, can be defined by the equation:

$$(f_1+f_2)(t_1, \hat{t}_2, \cdots, \hat{t}_n) = \begin{cases} f_1(2t_1, \hat{t}_2, \cdots, \hat{t}_n), & 0 \leqslant t_1 \leqslant 1/2 \\ f_2(2t_1 - 1, \hat{t}_2, \cdots, \hat{t}_n), & 1/2 \leqslant t_1 \leqslant 1. \end{cases}$$

Thus if $f_1 \in C^n(X, *, g_1)$ and $f_2 \in C^n(X, *, g_2)$, then $f_1 + g_1 f_2$ is defined and is a map of C^n of order $g_1 g_2$. The homotopy class of $f_1 + g_1 f_2$ depends only on the homotopy classes of f_1 and f_2. Thus the equation

$$[f_1; g_1] * [f_2; g_2] = [f_1 + g_1 f_2; g_1 g_2]$$

defines a rule of composition for maps of C^n of prescribed order. The set of homotopy classes of maps of C^n of prescribed order with this rule composition forms a group. This group was denoted by $\sigma_n(X, *, G)$ in [4].

3. The evaluation subgroups

Let (X, G, Π) be a transformation group, A be a pointed space and

$h:(A,*)\to(X,*)$ be a base point preserving map. Consider a continuous map $F: A\times C^n\to X$ such that

$$F(a,0,\hat{t}_2,\cdots,\hat{t}_n)=ha,\quad F(a,1,\hat{t}_2,\cdots,\hat{t}_n)=gha$$

where g is an element of G. If we define $f: C^n\to X$ by $f(t_1,\hat{t}_2,\cdots,\hat{t}_n)=F(*,t_1,\hat{t}_2,\cdots,\hat{t}_n)$, then we have $f(0,\hat{t}_2,\cdots,\hat{t}_n)=*$ and $f(1,\hat{t}_2,\cdots,\hat{t}_n)=g(*)$. Thus $[f;g]$ is an element of $\sigma_n(X,*,G)$. In this case, we say that F is an *affiliated map* of $[f;g]$ with respect to A of order g, or f is called *the trace* of F.

DEFINITION 3.1. The set of all elements $[f;g]\in\sigma_n(X,*,G)$ obtained in the above manner from some F will be denoted by $E_n^h(X,A,*,G)$.

Thus for every $[f;g]\in E_n^h(X,A,*,G)$, there is at least one map $F: A\times C^n\to X$ such that

$$F(a,0,\hat{t}_2,\cdots,\hat{t}_n)=ha,\quad F(a,1,\hat{t}_2,\cdots,\hat{t}_n)=gha$$

and $[f;g]=[F|_{*\times C^n},g]$. It is easy to see that $E_n^h(X,A,*,G)$ form a subgroup of $\sigma_n(X,*,G)$.

Let X^A be the space of continuous maps from A to X with compact open topology. Define $\Pi': X^A\times G\to X^A$ by $\Pi'(f,g)=gf$, where $gf(a)=\Pi(f(a),g)$. Then (X^A,G,Π') is a transformation group.

Let $p: X^A\to X$ be the evaluation map such that $p(f)=f(*)$. Then p is continuous and $p(gf)=(gf)(*)=g(f(*))=gp(f)$. Thus p is a homomorphism from (X^A,G) to (X,G). Therefore p induces a homomorphism

$$p_*:\sigma_n(X^A,h,G)\longrightarrow\sigma_n(X,*,G)$$

given by $p_*([f;g])=[p(f);g]$ for all n, where h is an element of X^A such that $h(*)=*$.

THEOREM 3.1. *Let A be locally compact and $p:(X^A,G)\to(X,G)$ be the evaluation homomorphism map. Then* $p_*(\sigma_n(X^A,h,G))=E_n^h(X,A,*,G)$.

Proof. Let $[u;g]\in\sigma_n(X^A,h,G)$. Then $u: C^n\to X^A$ is a continuous map such that $u(0,\hat{t}_2,\cdots,\hat{t}_n)=h$ and $u(1,\hat{t}_2,\cdots,\hat{t}_n)=gh$. Since A is locally compact, u give rise to the continuous associated map $\bar{u}: A\times C^n\to X$ such that $\bar{u}(a,t_1,\hat{t}_2,\cdots,\hat{t}_n)=u(t_1,\hat{t}_2,\cdots,\hat{t}_n)(a)$. Since

$$\bar{u}(*, t_1, \hat{t}_2, \cdots, \hat{t}_n) = u(t_1, \hat{t}_2, \cdots, \hat{t}_n)(*) = p(u(t_1, \hat{t}_2, \cdots, \hat{t}_n))$$
$$\bar{u}(a, 0, \hat{t}_2, \cdots, \hat{t}_n) = u(0, \hat{t}_2, \cdots, \hat{t}_n)(a) = ha$$
$$\bar{u}(a, 1, \hat{t}_2, \cdots, \hat{t}_n) = u(1, \hat{t}_2, \cdots, \hat{t}_n)(a) = gh(a)$$

$\bar{u}$ is an affiliated map of $[pu;g]$ with respect to A of order g. Thus $p_*([u;g]) = [pu;g] \in E_n^h(X, A, *, G)$.

Conversely, if $[f;g] \in E_n^h(X, A, *, G)$, then there exists an affiliated map $F : A \times C^n \to X$ such that

$$F(a, 0, \hat{t}_2, \cdots, \hat{t}_n) = ha$$
$$F(a, 1, \hat{t}_2, \cdots, \hat{t}_n) = gha$$
$$F(*, t_1, \hat{t}_2, \cdots, \hat{t}_n) = f(t_1, \hat{t}_2, \cdots, \hat{t}_n).$$

Its inverse associated map $\widetilde{F} : C^n \to X^A$ is a continuous map such that $\widetilde{F}(t_1, \hat{t}_2, \cdots, \hat{t}_n)(a) = F(a, t_1, \hat{t}_2, \cdots, \hat{t}_n)$. Thus we obtain

$$\widetilde{F}(0, \hat{t}_2, \cdots, \hat{t}_n)(a) = F(a, 0, \hat{t}_2, \cdots, \hat{t}_n) = ha$$
$$\widetilde{F}(1, \hat{t}_2, \cdots, \hat{t}_n)(a) = F(a, 1, \hat{t}_2, \cdots, \hat{t}_n) = gha.$$

Therefore $[\widetilde{F};g] \in \sigma_n(X^A, h, G)$ and $p_*([\widetilde{F};g]) = [p\widetilde{F};g] = [f;g]$.

REMARK. Let $A = \{*\} \subset X$ and $i : \{*\} \to X$ be the inclusion. Then $E_n^i(X, *, *, G) = \sigma_n(X, *, G)$ and $E_n^1(X, X, *, G) = E_n(X, *, G)$. (see [5])

THEOREM 3.2. *If A is a subspace of X and $i : (A, *) \to (X, *)$ is the inclusion, then $E_n^i(X, A, *, G)$ is a subgroup of $E_n^h(X, A, *, G)$ for any map $h : (A, *) \to (X, *)$ such that $h(A) \subset A$.*

Proof. Let $[f;g] \in E_n^i(X, A, *, G)$. Then there exists an affiliated map $F : A \times C^n \to X$ such that

$$F(a, 0, \hat{t}_2, \cdots, \hat{t}_n) = ia$$
$$F(a, 1, \hat{t}_2, \cdots, \hat{t}_n) = gia$$
$$F(*, t_1, \hat{t}_2, \cdots, \hat{t}_n) = f(t_1, \hat{t}_2, \cdots, \hat{t}_n).$$

If we define $H : A \times C^n \to X$ by

$$H(a, t_1, \hat{t}_2, \cdots, \hat{t}_n) = F(ha, t_1, \hat{t}_2, \cdots, \hat{t}_n),$$

then we obtain

$$H(a, 0, \hat{t}_2, \cdots, \hat{t}_n) = ha, \quad H(a, 1, \hat{t}_2, \cdots, \hat{t}_n) = gha,$$
$$H(*, t_1, \hat{t}_2, \cdots, \hat{t}_n) = F(*, t_1, \hat{t}_2, \cdots, \hat{t}_n) = f(t_1, \hat{t}_2, \cdots, \hat{t}_n).$$

Therefore $[f;g]\in E_n^h(X, A, *, G)$.

From now, $E_n^i(X, A, *, G)$ will be denoted by $E_n(X, A, *, G)$.

DEFINITION 3.2. A transformation group (X, G) is an *H-transformation group* if there exists an H-space structure $\mu : X\times X\to X$ such that $\mu(g*, y)=gy$ and $\mu(x, g*)=gx$.

EXAMPLE. (S^1, S^1, Π) is an H-transformation group, where $\Pi : S^1\times S^1\to S^1$ is given by $\Pi(e^{ix}, e^{iy})=e^{i(x+y)}$.

THEOREM 3.3. *If (X, G) is an H-transformation group, then $E_n(X, *, G)=E_n(X, A, *, G)=\sigma_n(X, *, G)$.*

Proof. Since $E_n(X, *, G)\leqslant E_n(X, A, *, G)\leqslant\sigma_n(X, *, G)$, it is sufficient to show that $\sigma_n(X, *, G)\leqslant E_n(X, *, G)$. Let μ be the H-transformation group structure and $[f;g]$ be an element of $\sigma_n(X, *, G)$. If we define $H : X\times C^n\to X$ by $H(x, t_1, \hat{t}_2, \cdots, \hat{t}_n)=\mu(x, f(t_1, \hat{t}_2, \cdots, \hat{t}_n))$, then

$$H(*, t_1, \hat{t}_2, \cdots, \hat{t}_n)=\mu(*, f(t_1, \hat{t}_2, \cdots, \hat{t}_n))=f(t_1, \hat{t}_2, \cdots, \hat{t}_n)$$
$$H(x, 0, \hat{t}_2, \cdots, \hat{t}_n)=\mu(x, *)=x$$
$$H(x, 1, \hat{t}_2, \cdots, \hat{t}_n)=\mu(x, g*)=g*.$$

Thus $[f;g]\in E_n(X, X, *, G)=E_n(X, *, G)$.

The fact that $E_n(X, *, G)\leqslant E_n(X, A, *, G)\leqslant\sigma_n(X, *, G)$ leads naturally to a quetion: Is there a topolgical transformation group pair (X, A, G) for which $E_n(X, *, G)<E_n(X, A, *, G)<\sigma_n(X, *, G)$?

EXAMPLE. Let $X=\{z\in R^2 : |z+1|=1 \text{ or } |z-1|=1\}$ and $A=S^1=\{z\in R^2 : |z-1|=1\}$. Let $h : X\to X$ be a homeomorphism given by $h(z)=-z$ and $G=\{h^n : n\in Z\}=Z_2$. If we define $\Pi : X\times G\to X$ by $\Pi(x, h^n)=h^n(x)$, then (X, G, Π) is a transformation group. Let us take $*=(0, 0)$ as the base point. Then we have $E_1(X, A, *, G)=Z$, $E_1(X, *, G)=0$ and $\sigma_1(X, *, G)=\pi_1(X, *)\times Z_2$.

F. Rhodes [4] showed that if λ is a path from $*_1$ to $*_2$, then λ induces an isomorphism $\lambda_* : \sigma_n(X, *_1, G)\to\sigma_n(X, *_2, G)$ given by $\lambda_*([f;g])=[\lambda^n\rho^n+f+g\lambda^n;g]$.

THEOREM 3.4. *If λ is a path from $*_1$ to $*_2$ in A, then λ induces an isomorphism $\lambda_* : E_n(X, A, *_1, G)\to E_n(X, A, *_2, G)$.*

Proof. Let $[f;g]$ be an element of $E_n(X, A, *_1, G)$. Then there exists an affiliated map H of order g with respect to A. That is, $H : A\times C^n \to X$ given by

$$
\begin{aligned}
&H(a, 0, \hat{t}_2, \cdots, \hat{t}_n) = a.\\
&H(a, 1, \hat{t}_2, \cdots, \hat{t}_n) = ga\\
&H(*_1, t_1, \hat{t}_2, \cdots, \hat{t}_n) = f(t_1, \hat{t}_1, \cdots, \hat{t}_n)
\end{aligned}
$$

Let λ be a path from $*_1$ to $*_2$. We want to show that $\lambda_*([f;g]) = [\lambda^n\rho^n + f + g\lambda^n; g]$ belongs to $E_n(X, A, *_2, G)$. Since $[H(*_2, \ );g]$ belongs to $E_n(X, A, *_2, G)$, it is sufficient to show that $[H(*_2, \ );g] = [\lambda^n\rho^n + f + g\lambda^n; g]$. If we define $Q : C^n \times I \to X$ by

$$
Q(t_1, \hat{t}_2, \cdots, \hat{t}_n, s) = \begin{cases} \lambda^n\rho^n(t_1, \hat{t}_2, \cdots, \hat{t}_n), & 0 \leqslant t_1 \leqslant (1-s)/2 \\ H(\lambda(s), \ (4t_1+2s-2)/(3s+1), \ \hat{t}_2, \cdots, \hat{t}_n), & \\ \qquad (1-s)/2 \leqslant t_1 \leqslant (s+3)/4 & \\ g\lambda^n(4t_1-3, \ \hat{t}_2, \cdots, \hat{t}_n), & (s+3)/4 \leqslant t_1 \leqslant 1 \end{cases}
$$

then Q is well defined and

$$
\begin{aligned}
&Q(t_1, \hat{t}_2, \cdots, \hat{t}_n, 0) = (\lambda^n\rho^n + f + g\lambda^n)(t_1, \hat{t}_2, \cdots, \hat{t}_n)\\
&Q(t_1, \hat{t}_2, \cdots, \hat{t}_n, 1) = H(*_2, t_1, \hat{t}_2, \cdots, \hat{t}_n)\\
&Q(0, \hat{t}_2, \cdots, \hat{t}_n, s) = *_2, \quad Q(1, \hat{t}_2, \cdots, \hat{t}_n, s) = g*_2.
\end{aligned}
$$

This means $[\lambda^n\rho^n + f + g\lambda^n; g] = [H(*_2, \ );g] \in E_n(X, A, *_2, G)$. Therefore $\lambda_*(E_n(X, A, *_1, G) \subset E_n(X, A, *_2, G)$. Since λ^{-1} is a path from $*_2$ to $*_1$ in A, we have $\lambda_*^{-1}(E_n(X, A, *_2, G)) \subset E_n(X, A, *_1, G)$. Therefore $\lambda_*(E_n(X, A, *_1, G) = E_n(X, A, *_2, G)$. Since λ_* is an isomorphism from $\sigma_n(X, *_1, G)$ to $\sigma_n(X, *_2, G)$, the proof was complete.

DEFINITION 3.3. Let (X, G) and (Y, H) be transformation groups. A category map $(\phi, \psi) : (X, G) \to (Y, H)$ is called a *pair category map* from (X, A, G) to (Y, B, H) if $\phi(A) \subset B$. A pair category map $(\phi, \psi) : (X, A, G) \to (Y, B, H)$ has a *right homotopy inverse* if there exists a pair category map $(\phi', \psi') : (Y, B, H) \to (X, A, G)$ such that $\phi\phi'$ is pair homotopic to 1_Y.

THEOREM 3.5. *If a pair category map* $(\phi, \psi) : (X, A, G) \to (Y, B, H)$ *has a right homotopy inverse* $(\phi', \psi') : (Y, B, H) \to (X, A, G)$, *then* $(\phi, \psi)_*(X, A, *, G)) \subset E_n(Y, B, \phi(*), H)$.

Proof. Choose a base point $*\in X$ and let $*_1=\phi'\phi(*)$. If $[f;g]\in E_n(X, A, *, G)$, then there exists an affiliated map $H: A\times C^n\to X$ of order g with respect to A. Since $\phi\phi'$ is pair homotopic to 1_Y, there is a homotopy $J: (Y, B)\times I\to(Y, B)$ such that $J(y, 0)=\phi\phi'(y)$ and $J(y, 1)=y$. If we define an affiliated map $T: B\times C^n\to Y$ of order ψg with respect to B by

$$T(b, t_1, \hat{t}_2, \cdots, \hat{t}_n)=\begin{cases} J(b, 1-3t_1), & 0\leqslant t_1\leqslant 1/3 \\ \phi H(\phi'(b), 3t_1-1, \hat{t}_2, \cdots, \hat{t}_n), & 1/3\leqslant t_1\leqslant 2/3 \\ \psi(g)J(b, 3t_1-2), & 2/3\leqslant t_1\leqslant 1, \end{cases}$$

then T is an affiliated map with trace $\tau=T(\phi(*), \ )$.

Let $\alpha: I\to Y$ be a path in Y given by $\alpha(t)=J(\phi(*), t)$. Then

$$\begin{aligned}\tau(t_1, \hat{t}_2, \cdots, \hat{t}_n)&=T(\phi(*), t_1, \hat{t}_2, \cdots, \hat{t}_n)\\ &=(\alpha^n\rho^n+\phi H(*_1, \ )+\psi(g)\alpha^n)(t_1, \hat{t}_2, \cdots, \hat{t}_n).\end{aligned}$$

Let λ be a path from $*$ to $*_1$ in A. Define a homotopy $Q: C^n\times I\to Y$ by

$$Q(t_1, \hat{t}_2, \cdots, \hat{t}_n, s)=\begin{cases} \phi\lambda^n\rho^n(2t_1, \hat{t}_2, \cdots, \hat{t}_n), & 0\leqq t_1\leqq(1-s)/2 \\ \phi H(\lambda(s), (4t_1+2s-2)/(3s+1), \hat{t}_2, \cdots, \hat{t}_n), & \\ \qquad (1-s)/2\leqq t_1\leqq(s+3)/4 & \\ \psi(g)\phi\lambda^n(4t_1-3, \hat{t}_2, \cdots, \hat{t}_n), & (s+3)/4\leqq t_1\leqq 1. \end{cases}$$

Then we obtain $[\phi\lambda^n\rho^n+\phi f+\psi(g)\phi\lambda^n;\psi(g)]=[\phi H(*_1, \ );\psi(g)]$.

$$\begin{aligned}\text{Moreover, } [\tau;\psi(g)]&=[\alpha^n\rho^n+H(*_1, \ )+\psi(g)\alpha^n;\psi(g)]\\ &=[\alpha^n\rho^n+\phi\lambda^n\rho^n+\phi f+\psi(g)\phi\lambda^n+\psi(g)\alpha^n;\psi(g)]\\ &=[(\phi\lambda^n+\alpha^n)\rho^n+\phi f+\psi(g)(\phi\lambda^n+\alpha^n);\psi(g)]\\ &=[(\phi\lambda^n+\alpha^n)\rho^n;e]*[\phi f;\psi(g)]*[\phi\lambda^n+\alpha^n;e]\end{aligned}$$

Therefore we have

$$\begin{aligned}[\phi f;\psi(g)]&=[(\phi\lambda^n+\alpha^n)\rho^n;e]^{-1}*[\tau;\psi(g)]*[\phi\lambda^n+\alpha^n;e]^{-1}\\ &=[\phi\lambda^n+\alpha^n;e]*[\tau;\psi(g)]*[(\phi\lambda^n+\alpha^n)\rho^n;e]\\ &=[\phi\lambda^n+\alpha^n+\tau+\psi(g)(\phi\lambda^n+\alpha^n)\rho^n;\psi(g)]\\ &=[(\phi\lambda+\alpha)\rho]_*([\tau, \psi(g)]).\end{aligned}$$

Since $(\phi\lambda+\alpha)\rho$ is a loop at $\phi(*)$ in B and $[\tau;\psi(g)]$ is an element of $E_n(Y, B, \phi(*), H)$, the theorem is proved by Theorem 3. 4.

DEFINITION 3. 4. Let (ϕ, ψ) be a pair category map from (X, A, G) to

(Y,B,H). We say that (ϕ,ψ) has a *left homotopy inverse* if there is a pair category map $(\phi',\psi'):(Y,B,H)\to(X,A,G)$ such that $\phi'\phi$ is pair homotopic to 1_X. Two pairs (X,A,G) and (Y,B,H) are said to be of the *same pair homotopy type* if there exist pair category maps (ϕ,ψ) : $(X,A,G)\to(Y,B,H)$ and $(\phi',\psi'):(Y,B,H)\to(X,A,G)$ such that ψ,ψ' are isomorphisms and $\phi\phi'$, $\phi'\phi$ are pair homotopic to 1_X, 1_Y respectively.

THEOREM 3.6. *Suppose $(\phi,\psi):(X,A,G)\to(Y,B,H)$ has a left homotopy inverse $(\phi',\psi'):(Y,B,H)\to(X,A,G)$. If $(\phi,\psi)_*([f;g])$ belongs to $E_n(Y,B,\phi(*),H)$, then $[f;g]$ belongs to $E_n(X,A,*,G)$.*

Proof. Let $[\phi f;\psi g]\in E_n(Y,B,\phi(*),H)$. Then there is an affiliated map $K:B\times C^n\to Y$ such that

$$\begin{aligned}&K(b,0,\hat{t}_2,\cdots,\hat{t}_n)=b\\&K(b,1,\hat{t}_2,\cdots,\hat{t}_n)=\psi(g)b\\&K(\phi(*),t_1,\hat{t}_2,\cdots,\hat{t}_n)=\phi f(t_1,\hat{t}_2,\cdots,\hat{t}_n)\end{aligned}$$

Since $\phi'\phi$ is homotopic to 1_X, there is a homotopy $J:(X,A)\times I\to(X,A)$ such that $J(x,0)=\phi'\phi(x)$, $J(x,1)=x$.

Define an affiliated map $T:A\times C^n\to X$ by

$$T(a,t_1,\hat{t}_2,\cdots,\hat{t}_n)=\begin{cases}J(a,1-3t_1), & 0\leqq t_1\leqq 1/3\\ \phi'K(\phi(a),\ 3t_1-1,\ \hat{t}_2,\cdots,\hat{t}_n), & 1/3\leqq t_1\leqq 2/3\\ J(ga,\ 3t_1-2), & 2/3\leqq t_1\leqq 1.\end{cases}$$

Then T is well defined, $T(a,0,\hat{t}_2,\cdots,\hat{t}_n)=a$ and $T(a,1,\hat{t}_2,\cdots,\hat{t}_n)=ga$. Thus T is an affiliated map of order g with respect to A.

Let $\alpha(t)=J(*,t)$ and τ be the trace of T at $*$. Then

$$\begin{aligned}\tau(t_1,\hat{t}_2,\cdots,\hat{t}_n)&=T(*,t_1,\hat{t}_2,\cdots,\hat{t}_n)\\&=(\alpha^n\rho^n+\phi'\phi f+J^n(g*,\ ))(t_1,\hat{t}_2,\cdots,\hat{t}_n).\end{aligned}$$

If we define $Q:C^n\times I\to X$ by

$$Q(t_1,\hat{t}_2,\cdots,\hat{t}_n,s)=\begin{cases}\alpha^n\rho^n(2t_1,\hat{t}_2,\cdots,\hat{t}_n), & 0\leqslant t_1\leqslant(1-s)/2\\ J(f((4t_1+2s-2)/(3s+1),\ \hat{t}_2,\cdots,\hat{t}_n),\ s), & \\ \qquad (1-s)/2\leqslant t_1\leqslant(s+3)/4 & \\ J^n(g*,\ 4t_1-3,\ \hat{t}_2,\cdots,\hat{t}_n), & (s+3)/4\leqslant t_1\leqslant 1.\end{cases}$$

then

$$[f;g]=[\alpha^n\rho^n+\phi'\phi f+J^n(g*,\ );g]$$
$$=[\tau;g]\in E_n(X,A,*,G).$$

F. Rhodes showed that if (X,G) and (Y,H) have the same homotopy type of transformation groups, then $(\phi,\psi)_*$ carries $\sigma_n(X,*,G)$ isomorphically onto $\sigma_n(Y,\phi(*),H)$. The following theorem is an extension of the Rhodes' result.

THEOREM 3.7. *If* $(\phi,\psi):(X,A,G)\rightarrow(Y,B,H)$ *is a pair homotopy equivalence homomorphism, then* $(\phi,\psi)_*$ *carries* $E_n(X,A,*,G)$ *isomorphically onto* $E_n(Y,B,\phi(*),H)$.

Proof. It is clear from Theorem 3.5 and Theorem 3.6.

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