Comm. Korean Math. Soc. 1 (1986)

A REMARK ON A PROJECTIVE COVER OF A MODULE

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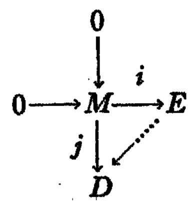
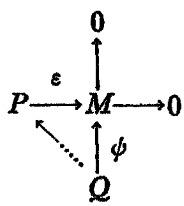
In this short note, we point an error which appeared in pp. 74-75 of Rotman's book, an introduction to homological algebra. He characterized there injective envelopes as follows: A monomorphism $i: M \to E$, where E is injective, is an injective envelope if and only if a monic dashed arrow always exists below



whenever j is an imbedding of M into an injective D.

Dualizing the notion of an injective envelope, he defined a *projective cover* of a module M iff it is an epimorphism $\varepsilon: P \to M$, where P is projective, so that an epic dashed arrow always exists below



whenever ϕ is an epimorphism from a projective Q.

He also characterized an epimorphism $\varepsilon: P \to M$, where P is projective, is a projective cover if and only if Ker ε is supperfluous. But, we recalled that a projective cover of a module M is an epimorphism $\varepsilon: P \to M$, where P is projective, such that Ker ε is superfluous, in any standard books in theory of rings and modules.

In this short remark, we show that the above characterization of a projective cover is not always true in the following proposition.

PROPOSITION. *An epimorphism* $\varepsilon : P \to M$, *where* P *is projective, is a projective cover (in sense of Rotmann) if* Ker ε *is superfluous. But its converse is not always true.*

Proof. Consider given a diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ P & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ & {\scriptstyle\phi}\nwarrow & \uparrow{\scriptstyle\psi} & & \\ & & Q & & \end{array}$$

where ψ is any epimorphism from a projective Q.

Since Q is projective, there exists an homomorphism ϕ such that $\varepsilon\phi=\psi$. Since Ker $\varepsilon+\mathrm{Im}\,\phi=P$ and Ker ε is superfluous, it follows that ϕ is epic. To see that its converse is not always true, let $\varepsilon : \boldsymbol{Z}\to\boldsymbol{Z}/2\boldsymbol{Z}$ be the canonical epimorphism as $\boldsymbol{Z}$-modules. Then it is known that Ker ε is not superfluous.

Consider given a diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ \boldsymbol{Z} & \xrightarrow{\varepsilon} & \boldsymbol{Z}/2\boldsymbol{Z} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle\psi} & & \\ & & Q & & \end{array}$$

where Q is an $\boldsymbol{Z}$-projective and ψ is any epimorphism from

Q to $\boldsymbol{Z}/2\boldsymbol{Z}$. Since Q is $\boldsymbol{Z}$-free, we may assume that $Q=\coprod \boldsymbol{Z}_\alpha$, where $\boldsymbol{Z}_\alpha=\boldsymbol{Z}$ for each α. Let $u_\alpha : \boldsymbol{Z}_\alpha\to\coprod\boldsymbol{Z}_\alpha$ be the αth injection. By assumption ψ is an epimorphism, and so there exists α such that $\psi u_\alpha=\varepsilon$.

For each α define $\phi_\alpha : \boldsymbol{Z}_\alpha\to\boldsymbol{Z}$ as follows:

$$\phi_\alpha=\begin{cases}1 & \text{if } \psi u_\alpha=\varepsilon \\ 0 & \text{if } \psi u_\alpha=0,\end{cases}$$

where 1 is the identity map on $\boldsymbol{Z}$ and 0 is the trivial homomorphism on $\boldsymbol{Z}$.

Thus $\varepsilon\phi_\alpha=\psi u_\alpha$ for all α. Let $\phi=\coprod\phi_\alpha$ be the coproduct map of the family $\{\phi_\alpha\}$. Then it is easy to show that ϕ is an epimorphism by using the existence of α such that $\psi u_\alpha=\varepsilon$. Moreover, we have

$$\psi u_\alpha=\varepsilon\phi_\alpha=\varepsilon\phi u_\alpha$$

for all α. Hence we have $\psi=\varepsilon\phi$.

Reference

1. J.J. Rotman, *An introduction to homological algebra*, Academic Press, New York, 1979.

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