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REMARKS ON THE FUNCTORS TOR AND EXT

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In the homology theory, to make new functors and find some relations between functors are very important ([4], [7]). The general functors in the homological algebras are functors Tor and Ext which are completely related to each other ([2], [3]).

If we can prove that

$$\mathrm{Ext}_{\Lambda}^{n-q}(K, A) = \mathrm{Tor}_q^{\Lambda}(K, A) \quad (0 \leqslant q \leqslant n) \qquad (*)$$

under some conditions, where $n(\geqslant 0)$ is a fixed integer and Λ is a commutative algebra, then there may exist many relations among algebras ([5], [6]).

In this paper, we shall prove above (*) when Λ is a polynomial ring in Lemma 2.

The purpose of this paper is to apply Lemma 2 to Theorem 6 which is in [1]. This was suggested by referee. I would like to thank him.

Let K be a field, and let $\Lambda = K[x_1, \cdots, x_n]$ be the polynomial ring with n-letters $x_1, \cdots, x_n$ and coefficient ring K. Let I_k be the ideal of Λ generated by $\{x_1, \cdots, x_k\}$, where $0 \leqslant k \leqslant n$. If $k=0$ then $I_k = I_0 = 0$. We put $\Gamma = \boldsymbol{Z}[x_1, \cdots, x_n]$ which is the polynomial ring over the integers $\boldsymbol{Z}$. Then Λ is regarded as a Γ-module. We shall denote J_k the ideal of Γ generated by $\{x_1, \cdots, x_k\}$ $(0 \leqslant k \leqslant n)$. Then it is obvious that $I_k = \Lambda J_k$.

Suppose a Γ-module M. We want to make a complex over the module M/J_nM. In order to do this we consider the exterior algebra $E(x_1, \cdots, x_n)$ with coefficients in $\boldsymbol{Z}$ as follows:

$E(x_1, \cdots, x_n) = E_0(x_1, \cdots, x_n) \oplus \cdots \oplus E_n(x_1, \cdots, x_n)$ where for $q (0 \leqslant q \leqslant n)$ $E_q(x_1, \cdots, x_n)$ is a $\boldsymbol{Z}$-free module with basis $\{x_{i_1}, \cdots, x_{i_q} \mid 1 \leqslant i_1 < \cdots < i_q \leqslant n\}$ (where x_i and x_j $(1 \leqslant i, j \leqslant n)$ are commutative). In particular $E_0(x_1, \cdots, x_n) = \boldsymbol{Z}$.

Note that if we regard x_i $(1 \leqslant i \leqslant n)$ as an element with degree 1 then ① Λ is a graded K-module, ② Γ is a graded $\boldsymbol{Z}$-module, ③ $E(x_1, \cdots, x_n)$

is a graded $\boldsymbol{Z}$-module.

Since

$$M \otimes_Z E_0(x_1, \cdots, x_n) = M$$

we have the natural epimorphism

$$\varepsilon : M = M \otimes_Z E_0(x_1, \cdots, x_n) \longrightarrow M/MJ_n.$$

We consider the complex X over M/J_nM such that

$$X : 0 \to M \otimes E_n(x, \cdots, x_n) \xrightarrow{d_n} \cdots \longrightarrow M \otimes E_1(x_1, \cdots, x_n) \xrightarrow{d_1} M \otimes E_0(x_1, \cdots, x_n) \xrightarrow{\varepsilon} M/MJ_n$$

where

$$d_i : M \otimes E_i(x_1, \cdots, x_n) \longrightarrow M \otimes E_{i-1}(x_1, \cdots, x_n) \quad (1 \leqslant i \leqslant n)$$

in defined by

$$d_i(m \otimes x_{p_1}, \cdots, x_{p_i}) = \sum_{1 \leq j \leq i} (-1)^{j+1} (m x_{p_j}) \otimes x_{p_1} \cdots \hat{x}_{p_j} \cdots x_{p_i}$$

and $\otimes = \otimes_Z$. Of course, $1 \leqslant p_1 < \cdots < p_j \leqslant n$ and $\hat{x}_{p_j}$ indicates that x_{p_j} is to be omitted. Then it is easy to prove that ① $\varepsilon d_1 = 0$ and ② $d_{i-1} d_i = 0$ $(2 \leqslant i \leqslant n)$. Therefore, X is a complex over a Γ-module M/MJ_n. In our situation, the following holds ([1]).

1° If M satisfies

$$(m \in M, \ m x_k \in MJ_{k-1}) \Longrightarrow (m \in MJ_{k-1}) \quad (1 \leqslant k \leqslant n)$$

then X is acyclic.

We shall return to discuss $\Lambda = K[x_1, \cdots, x_n]$. Since Λ is a Γ-module by the same way as above

$$X : 0 \longrightarrow \Lambda \otimes E_n(x_1, \cdots, x_n) \longrightarrow \cdots \longrightarrow \Lambda \otimes E_0(x_1, \cdots, x_n) \longrightarrow \Lambda/\Lambda J_n = Q_\Lambda$$

is a complex over $Q_\Lambda = K$, where $\otimes = \otimes_Z$. (Note that since $\Lambda J_n = I_n =$ the ideal of Λ generated by $\{x, \cdots, x_n\}$ we have $\Lambda/\Lambda J_n = \Lambda/I_n = K$.). It follows that the Γ-module Λ satisfies the hypothesis of 1°. For each $0 \leqslant i \leqslant n$

$$\Lambda \otimes E_i(x_1, \cdots, x_n) \quad (\otimes = \otimes_Z)$$

is a Λ-free module with a basis

$$\{1\times x_{p_1}\cdots x_{p_i} \mid 1\leqslant p_1<\cdots<p_i\leqslant n\},$$

and thus it is a Λ-projective module. Hence, by 1° the complex

$$X: 0\rightarrow\Lambda\otimes E_n(x_1,\cdots,x_n)\longrightarrow\cdots\longrightarrow\Lambda\otimes E_1(x_1,\cdots,x_n)\longrightarrow\Lambda\xrightarrow{\varepsilon}K$$

is a Λ-projective resolution over Q_Λ.

LEMMA 2. *In the above situation*

$$\mathrm{Tor}_q^\Lambda(K,A)\cong\mathrm{Ext}_\Lambda^{n-q}(K,A),\quad(0\leqslant q\leqslant n)$$

where A *is an arbitrary* Λ*-module.*

Proof. As before

$$X_q=\Lambda\otimes E_q(x_1,\cdots,x_n)\cong\Lambda\oplus\cdots\oplus\Lambda\ \ ({}_nC_q\text{-times}).$$

On the other hand,

$$\begin{aligned}\mathrm{Hom}_\Lambda(X_{n-q},\Lambda)&=\mathrm{Hom}_\Lambda(\Lambda\otimes E_{n-q}(x_1,\cdots,x_n),\ \Lambda)\\&\cong\mathrm{Hom}_\Lambda(\Lambda\oplus\cdots\oplus\Lambda({}_nC_{n-q}\text{ times}),\ \Lambda)\\&\cong\Lambda\oplus\cdots\oplus\Lambda({}_nC_{n-q}\text{ times}).\end{aligned}$$

Since ${}_nC_q={}_nC_{n-q}$ we have the isomorphism

$$\phi_q: X_q\xrightarrow{\cong}\mathrm{Hom}_\Lambda(X_{n-q},\ \Lambda).$$

In consequence, we have got a Λ-projective resolution over K such that

$$\mathrm{Hom}_\Lambda(X,\Lambda): 0\rightarrow\mathrm{Hom}_\Lambda(X_0,\Lambda)\rightarrow\cdots\rightarrow\mathrm{Hom}_\Lambda(X_n,\Lambda)\rightarrow\Lambda/I_n=K.$$

Furthermore, we can prove that

$$\mathrm{Hom}_\Lambda(X,\Lambda)\otimes_\Lambda A\longrightarrow\mathrm{Hom}_\Lambda(X,A)$$

is isomorphic as follows.

$$\begin{aligned}\mathrm{Hom}_\Lambda(X_{n-q},A)&\cong\mathrm{Hom}_\Lambda(\Lambda\oplus\cdots\oplus\Lambda\ \ ({}_nC_{n-q}\text{ times}),\ A)\\&\cong\mathrm{Hom}_\Lambda(\Lambda,A)\oplus\cdots\oplus\mathrm{Hom}_\Lambda(\Lambda,A)\ \ ({}_nC_{n-q}\text{ times})\\&\cong A\oplus\cdots\oplus A\ \ ({}_nC_{n-q}\text{ times}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{Hom}_\Lambda(X_q, \Lambda)\otimes_\Lambda A &\cong \mathrm{Hom}_\Lambda\ (\Lambda\oplus\cdots\oplus\Lambda\,({}_nC_q\text{-times}),\ \Lambda)\otimes_\Lambda A\\ &\cong \Lambda\otimes_\Lambda A\oplus\cdots\oplus\Lambda\otimes_\Lambda A\ \ ({}_nC_q\text{-times})\\ &\cong A\oplus\cdots\oplus A\ \ ({}_nC_q\text{-times}).\end{aligned}$$

Thus, by using the homology functor H we have

$$H_q(\mathrm{Hom}(X, \Lambda)\otimes_\Lambda A)\cong H^{n-q}(\mathrm{Hom}_\Lambda(X, A)).$$

That is,

$$\mathrm{Tor}_q{}^\Lambda(K, A)\cong \mathrm{Ext}_\Lambda{}^{n-q}(K, A)$$

for all $q=0, 1, \cdots, n$. ///

Let Σ be a ring with unity and let I be a proper two-side ideal of Σ. Then there is the augmentation epimorphism

$$\varepsilon : \Sigma \longrightarrow \Sigma/I=Q_\Sigma.$$

Let A be a right Σ-module. Then it follows from the exact sequence

$$0\longrightarrow I\longrightarrow \Sigma \overset{\varepsilon}{\longrightarrow} Q_\Sigma \longrightarrow 0$$

that

$$A\otimes_\Sigma Q_\Sigma \cong A/AI.$$

DEFINITION 3. We assume the above situation.

(1) If $A=0$ or $A\otimes_\Sigma Q_\Sigma\neq 0$ then A is said to be *proper*. Thus, every free Σ-module is proper.

(2) For a subset $N\subset A$ let F_N be the free Σ-module generated by N. Then there exists the natural homomorphism $\phi_N : F_N\rightarrow A$. Put Ker $\phi_N=R_N$ and Coker $\phi_N=L_N$. Then we have the exact sequence of right Σ-modules:

$$0\longrightarrow R_N\longrightarrow F_N\overset{\phi_N}{\longrightarrow} A\longrightarrow L_N\longrightarrow 0.$$

Take a subset $M\subset A$. If for every subset N of M the modules R_N and L_N are proper then M is said to be *faithful.*

We have the following fact ([1]):

4° For a graded ring Σ, every graded Σ-module is proper and every set of homogeneous elements is faithful.

Again, we suppose the ring $\Lambda=K[x_1, \cdots, x_n]$. As before, Λ is graded ring. Thus, by 4° when we regard Λ as a right Λ-module, Λ is a faithful right Λ-module. Moreover K is a faithful subset of K and $K\Lambda=\Lambda$, i.e., K is a faithful subset generating Λ. In particular, $\{1\}$ (1 is the identity of K) is a K-base of K and it is a Λ-base of Λ. Of course, $\{1\}$ is a faithful subset of Λ ($L_{\{1\}}=R_{\{1\}}=0$) and $\mathrm{Tor}_1{}^{\Lambda}(\Lambda, K)=0$. The inverse of these holds as follows.

LEMMA 5. *For a Λ-module A and its faithful subset M, if the image of M in $A\otimes_\Lambda K\cong A/AI_k$ ($K=\Lambda/I_k$) generates $A\otimes_\Lambda K$ as a K-module, then M generates A. Furthermore, if $\mathrm{Ext}_\Lambda{}^{n-1}(K, A)=0$ and the images of M in $A\otimes_\Lambda K$ form a K-base for $A\otimes_\Lambda K$ then M is a Λ-base for A.*

Proof. Note that K is a Λ-module with Λ-actions

$$x_iK=Kx_i=0, \quad i=1, \cdots, n.$$

From the exact sequence

$$0\longrightarrow R_M\longrightarrow F_M\xrightarrow{\phi_M} A\longrightarrow L_M\longrightarrow 0,$$

we have the exact sequence

$$R_M\otimes_\Lambda K\longrightarrow F_M\otimes_\Lambda K\xrightarrow{\phi_M\otimes 1_K} A\otimes_\Lambda K\longrightarrow L_M\otimes_\Lambda K\longrightarrow 0.$$

Since the images of M in $A\otimes_\Lambda K$ generates $A\otimes_\Lambda K$ as a right K-module, $\phi_M\otimes 1_K$ is an epimorphism. Therefore $L_M\otimes_\Lambda K=0$. By our assumption, since L_M is a proper Λ-module we have $L_M=0$ (see (2) of Definition 3). Therefore $\phi_M : F_M\rightarrow A$ is an epimorphism. That is, M generates A.

By Lemma 2

$$\mathrm{Tor}_1{}^{\Lambda}(A, K)\cong\mathrm{Tor}_1{}^{\Lambda}(K, A)\cong\mathrm{Ext}_\Lambda{}^{n-1}(K, A)=0.$$

From the exact sequence

$$0\longrightarrow R_M\longrightarrow F_M\xrightarrow{\phi_M} A\longrightarrow 0 \qquad (**)$$

we have the exact sequence

$$\cdots\longrightarrow\mathrm{Tor}_1{}^{\Lambda}(K, A)\longrightarrow R_M\otimes_\Lambda K\longrightarrow F_M\otimes_\Lambda K\xrightarrow{\phi_M\otimes 1_K} A\otimes_\Lambda K,$$

i.e.,

$$\cdots\longrightarrow 0\longrightarrow R_M\otimes_\Lambda K\longrightarrow F_M\otimes_\Lambda K\xrightarrow{\phi_M\otimes 1_K} A\otimes_\Lambda K.$$

By the assumption, since the image of M in $A\otimes_\Lambda K$ is a K-base for $A\otimes_\Lambda K$, $\phi_M\otimes 1_K$ is an isomorphism, and thus $R_M\otimes_\Lambda K=0$. Also, since R_M is proper we have $R_M=0$. Hence, by the exact sequence (**)

$$\phi_M : F_M\longrightarrow A$$

is an isomorphism. Therefore, M is a Λ-base for A. ///

THEOREM 6. *Let A be a Λ-module. If* $\mathrm{Ext}_\Lambda{}^{n-1}(K, A)=0$ *then every faithful subset Λ-generating of A contains a Λ-base for A. Thus, if A is generated by a finite set then A is a Λ-free module.*

Proof. Note that Λ is a local ring with the maximal ideal I_n. Moreover, $\Lambda=K[x_1, \cdots, x_n]$ is a Noetherian ring since K is a field. Therefore, every finite subset of A is a faithful subset of A ([1]). Thus, if the first part of our theorem is proved then it is obvious that the second part of our theorem holds.

Let M be a faithful Λ-generating subset of A. Then the image of M in $A\otimes_\Lambda K$ generates $A\otimes_\Lambda K$ as a K-module, i.e., as a K-vector space. If $\{m_i \mid i\in B,\ B$ is an indexing set$\}$ then we can choose a K-base for $A\otimes_\Lambda K$ in the set $\{m_i\otimes 1 \mid i\in B\}$ i.e., if $\{m_{i_j}\otimes 1 \mid i_j\in B\}$ M is a K-base for $A\otimes_\Lambda K$ then elements $m_{i_j}\otimes 1$ is K-independent. Thus, by Lemma 5 $\{m_{i_j}\otimes 1 \mid i_j\in B\}\subset$M is a K-base for A. Since a finite subset of A is faithful, in this case, A is a free Λ-module. ///

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