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CR STRUCTURES AND THE MIZOHATA TYPE OPERATORS

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1. Introduction

If we compare the Mizohata Operator

$$M=\frac{1}{2}\ \frac{\partial}{\partial x}-ix\frac{\partial}{\partial u}$$

with the Lewy operator

$$L=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)-i(x+iy)\frac{\partial}{\partial u}$$
$$=\partial_{\bar{z}}-iz\partial_u,$$

we can notice a remarkable similitude between them; for example, by setting $y=0$ we can get the Mizohata operator from Lewy's one, a solution of $Mv=0$ is $v=u+ix^2$ while that of $Lw=0$ is $w=u+i|z|^2$ and so on.

Until now, however, it seems that there has been no unified theory governing the Mizohata operator and the Lewy operator simultaneously.

In this paper we are attempting to develop a theory which will clarify a link between these two operators. Our standpoint is to regard Lewy operator L as an induced CR operator on a real hypersurface M^3 in C^2 and to get the Mizohata operator M as a push-forward $\pi_*(L)$ of L by a certain projection π of M. Through these points of view, we apply exactly the same arguments developed in Jacobowitz and Treves [1] for the Lewy type operators to derive that certain class of complex vector fields of the Mizohata type

$$\tilde{M}=\frac{1}{2}(1+iG_u)\frac{\partial}{\partial x}-i\left(x+\frac{1}{2}G_x+f\right)\frac{\partial}{\partial u}$$

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has the property that if $\widetilde{M}v=0$, then v is constant in a neighborhood of the origin.

2. CR Structures and the Mizohata Type Operators

We start with a real hypersurface M^3 in C^2. Let TM^3 and TC^2 be the tangent spaces to M^3 and C^2, respectively. The complexified tangent spaces $C\otimes TC^2$ has the decomposition $T_{1,0}+T_{0,1}$ into holomorphic and anti-holomorphic vectors. Using the usual coordinates (z, w) in C^2 we have that $T_{1,0}$ is spanned by $\{\partial_z, \partial_w\}$ and $T_{0,1}$ by $\{\partial_{\bar{z}}, \partial_{\bar{w}}\}$. Set

$$V=(C\otimes TM)\cap T_{0,1}.$$

We note that $V\cap\bar{V}=\{0\}$ while $\overline{C\otimes TM}=C\otimes TM$. Thus V is always a one dimensional subbundle of $C\otimes TM$. Let L span V and think of L as a partial differential operator. We define L as an *induced CR operator* on M.

We will be especially interested in manifolds $M^3\subset C^2$ given as graphs of the form

$$M=\{(z,\ u+iv)\,|\,v=H(z, \bar{z}, u)\}.$$

As local coordinates on such manifold we may take $(z, \bar{z}, u)$, or equivalently, (x, y, u) where $z=x+iy$. The CR operator must annihilate z and $w=u+iH(z, \bar{z}, u)$. Thus it has the form

$$L=(1+iH_u(z, \bar{z}, u))\partial_{\bar{z}}-iH_{\bar{z}}(z, \bar{z}, u)\partial_u$$

Now if we fix some point $p\in M$, we can then find a complex affine map ϕ so that $\phi(p)$ is the origin and $\{(z, w)\,|\,w=0\}$ is the tangent plane at the origin to $\phi(M)$. If we call this new manifold M, we see that M has the form

$$M=\{(z, w)\,|\,w=u+iv \text{ with } v=H(z, \bar{z}, u)\}$$

where H and its first derivatives are zero at the origin and L has the form

$$L=(1+iH_u(z, \bar{z}, u))\partial_{\bar{z}}-iH_{\bar{z}}(z, \bar{z}, u)\partial_u.$$

If we assume further that

(A_1) *$L, \bar{L}$ and $[L, \bar{L}]$ are linearly independent at the origin,* then, following the argument of Jacobowitz and Treves [1], we may take

$$H=|z|^2+G(z,\bar{z},u)$$

with $G=G_u=G_z=G_{\bar{z}}=G_{zz}=G_{z\bar{z}}=G_{\bar{z}\bar{z}}=0$ at the origin.

Under the assumption (A_1), L now has the form

$$L=(1+iG_u)\partial_{\bar{z}}-i(z+G_{\bar{z}})\partial_u$$

where

$$G=\beta zu+\overline{\beta z}u+cu^2+\cdots \quad (c: \text{real}).$$

When $G=0$, L becomes *the Lewy operator.*

From now on we shall use local coordinates (x, y, u) instead of $(z, \bar{z}, u)$. We denote by π the projection of M to (x, u, v) space defined by

$$\pi(x, y, u)=(x, 0, u).$$

We see that $\pi(M)$ is a 2-dimensional hypersurface in R^3 given by

$$\pi(M)=\{(x,\ u+iv)\,|\,v=x^2+F(x, 0, u)\}$$

where $F(x, y, u)=G(z, \bar{z}, u)$. We also have

$$\pi_*(L)=\frac{1}{2}(1+iF_u(x, 0, u))\frac{\partial}{\partial x}-i(x+F_{\bar{z}}(x, 0, u))\frac{\partial}{\partial u}.$$

When $F=0$, $\pi_*(L)$ becomes *the Mizohata operator.*

Now we define for $\lambda=\mu+i\eta$ the set

$$\Gamma(\lambda)=\{(x, \lambda)\,|\,\eta=x^2+F(x, 0, \mu)\}.$$

THEOREM 2.1. *There is a smooth curve γ_s in the λ-plane given by $\eta=\eta(\mu)$ such that*

1) *for $\eta<\eta(\mu)$, $\Gamma(\lambda)$ is empty,*
2) *for $\eta=\eta(\mu)$, $\Gamma(\mu+i\eta(\mu))$ is a point which varies smoothly in μ, and*
3) *for $\eta>\eta(\mu)$, $\Gamma(\mu+i\eta)$ is the set of two points which varies smoothly in λ.*

Proof. Let

$$A(x, u)=x^2+F(x, 0, u).$$

We may write $A(x, u)$ as

$$A(x, u) = x^2 + b(u)x + c(u) + x^3 K(x, u).$$

Thus for each fixed u, the minimum of F occurs at a point $x(u)$ which varies smoothly in u. We set

$$\eta(\mu) = A(x(\mu), \ \mu).$$

Then 1) and 2) follows.

We also have

$$A(x, u) = \eta(u) + Q(x, u) + \cdots$$

where Q is a positive definite quadratic in $x_1 = x - x(u)$. It follows that the level set $A = \eta$ for $\eta > \eta(u)$ consists of two points varying smoothly with respect to u. This establishes 3). Q.E.D.

Let us denote the curve $\Gamma(\mu + i\eta(\mu))$ by Γ_s. Take C to be a smooth simple closed curve in $\pi(M)$ which is close to the origin and which does not intersect $x=0$ or the curve Γ_s. Then for each point λ of C we can make $\Gamma(\lambda)$. The union of these sets $\Gamma(\lambda)$ is the disjoint union of two smooth simple closed curves on $\pi(M)$ separated by Γ_s which we denote by $S = C_1 \cup C_2$ with $C = C_1$.

Let $T = T_1 \cup T_2$ be the (open) region bounded by S and Ω be an arc connected set containing $\overline{T}$ and a part of the curve Γ_s. Further we assume that if Ω contains some point of $\Gamma(\lambda)$, then it contains the other point of $\Gamma(\lambda)$.

Under the above assumptions we have the following:

For any set $\Gamma(\lambda)$ which is in $\Omega \backslash T$ or on S there is a smooth family $T(\lambda(t))$ with

1) $\Gamma(\lambda(0)) = \Gamma(\lambda)$,
2) $\Gamma(\lambda(1)) \in \Gamma_s$, and
3) $\Gamma(\lambda(t)) \in \Omega \backslash T$ for $0 \leqq t \leqq 1$.

From now on we assume further that

(A_2) *$F(x, y, u)$ is a function of x and u alone; that is,*

$$F(x, y, u) = F(x, u)$$

Then $\pi_*(L)$ has the form

$$\pi_*(L) = \frac{1}{2}(1 + iF_u)\frac{\partial}{\partial x} - i\left(x + \frac{1}{2}F_x\right)\frac{\partial}{\partial u}.$$

For any $\lambda=u+i\eta$ where $\Gamma(\lambda)$ is nonempty we set

$$\Gamma(\lambda)=\{(x_1, u),\quad (x_2, u)\}$$

with $\lambda=u+i\eta$. Here (x, u) is the local coordinates for $\pi(M)$.

THEOREM 2.2. (*Local Constancy Principle*). *Let h be a C^1 function on $\pi(M)$.*

1) *If $\pi_*(L)h=0$ in some open set $\mathfrak{D}$, then*

$$h(x_1, u)-h(x_2, u)$$

is a holomorphic function of $\lambda=u+\lambda\eta$ with $\eta>\eta(u)$.

2) *If $\pi_*(L)h=0$ on $\Omega\backslash T$ and $\Gamma(\lambda)\subset\Omega\backslash T$, then*

$$h(x_1, 0, u)-h(x_2, 0, u)=0.$$

3) *If $\pi_*(L)h=0$ on $\Omega\backslash T$, then*

$$\int_S h\; dw=0 \quad (w=\lambda).$$

Here $S=C_1\cup C_2$ is oriented so that its projection on (u, x) plane is counter clockwise.

Proof. 1) From $\pi_*(L)h=0$, we have

$$\frac{1}{2}(1+iF_u)\frac{\partial h}{\partial x}-i\left(x+\frac{1}{2}F_x\right)\frac{\partial h}{\partial u}=0.$$

As

$$w=u+iv=u+i(x^2+F(x, u)),$$

we have

$$\frac{1}{2}(1+iF_u)(h_w w_x+h_{\bar{w}}\bar{w}_x)-i\left(x+\frac{1}{2}F_x\right)(h_w w_u+h_{\bar{w}}\bar{w}_u)=0.$$

Or, equivalently,

$$\left[\frac{1}{2}(1+iF_u)w_x-i\left(x+\frac{1}{2}F_x\right)w_u\right]h_w+\left[\frac{1}{2}(1+iF_u)\bar{w}_x-i\left(1+\frac{1}{2}F_x\right)\bar{w}_u\right]h_{\bar{w}}=0$$

But

$$\pi_*(L)w=\frac{1}{2}(1+iF_u)w_x-i\left(x+\frac{1}{2}F_x\right)w_u=0$$

and

$$\pi_*(L)\bar{w}=\frac{1}{2}(1+iF_u)\bar{w}_x-i\left(x+\frac{1}{2}F_x\right)\bar{w}_u=-i(2x+F_x).$$

Thus $\pi_*(L)\bar{w}\neq 0$ for $\eta>\eta(u)$. This implies $h_{\bar{w}}=0$ for $\lambda=w=u+i\eta$ with $\eta>\eta(u)$.

2) Let

$$\Omega_0=\{\lambda\,|\,\Gamma(\lambda)\text{ nonempty and }\Gamma(\lambda)\subset\Omega\backslash T\}.$$

We note that Ω_0 is arc connected and contains a part of γ_s. The function $h(x_1,u)-h(x_2,u)$ is holomorphic in the interior of Ω_0, is a C^1 function on Ω_0 and vanishes on $\gamma_s\cap\Omega_0$. Therefore, $h(x_1,u)-h_2(x_2,u)$ vanishes identically on Ω_0.

3) We note that the projection of C_1 and C_2 on (u,v) plane is identical but of opposite orientation. Therefore, we have

$$\begin{aligned}\int_S h\,dw&=\int_{C_1\cup C_2} h\,dw\\&=\int_{C_1} h(x_1,u)\,dw+\int_{C_2} h(x_2,u)\,dw\\&=\int_{C_1}(h(x_1,u)-h(x_2,u))\,dw\\&=0\end{aligned}$$

as $h(x_1,u)=h(x_2,u)$ by 2). Q.E.D.

REMARK. It should be noted that in the case of Mizohata operator, i.e., when $F=0$ Treves [5] used the fact

$$\int_{C_2} h\;dw=0$$

for h which satisfies $\pi_*(L)h=0$ in Ω. This fact follows from 1). In our arguments in the next section, however, 3) is sufficient.

THEOREM 2.3. *Let h be any C^1 function on an open set $\mathfrak{D}\subset\pi(M)$ where $\mathfrak{D}$ has a smooth boundary. Then*

$$\int_{\partial\mathscr{D}} h\ dw = 2\int_{\mathscr{D}} \pi_*(L)h\ dxdu$$

Proof. As $w=u+i(x^2+F(x,u))$, we have

$$dw=(1+iF_u)du+i(2x+G_x)dx.$$

Therefore,

$$\begin{aligned}\int_{\partial\mathscr{D}} h\ dw &= \int_{\partial\mathscr{D}} h[(1+iF_x)du+i(2x+F_x)dx]\\ &=\iint_{\mathscr{D}}[h_x(1+iF_u)+(1+iG_u)_x h]dx\wedge du\\ &\quad-\iint_{\mathscr{D}}[h_u i(2x+F_x)+i(2x+F_x)_u h]dx\wedge du\\ &=\iint_{\mathscr{D}}[h_x(1+iF_u)+iF_{ux}h]dx\wedge du\\ &\quad-\iint_{\mathscr{D}}[h_u i(2x+F_x)+iF_{xu}h]dx\wedge du\\ &=\iint_{\mathscr{D}}[(1+iF_u)h_x-i(2x+F_x)h_u]dx\wedge du\\ &=2\iint_{\mathscr{D}}\left[\frac{1}{2}(1+iF_u)\frac{\partial}{\partial x}-i\left(x+\frac{1}{2}F_x\right)\frac{\partial}{\partial u}\right]h\ dx\wedge du\\ &=2\iint_{\mathscr{D}}\pi_*(L)h\ dxdu \qquad \text{Q.E.D.}\end{aligned}$$

3. Local Constancy of the Solution

In this last section we prove that there exists a perturbation of $\pi_*(L)$ to $\pi_*(L)+fQ$ so that the partial differential equation

$$(\pi_*(L)+fQ)v=0$$

has only constant solutions in a neighborhood of the origin. This generalizes the results of Nirenberg [4] and Treves [5] that have proved the same results for the Mizohata operator M.

As in the section 2 we let L be an induced CR operator on a real hypersurface M^3 in C^2. We assume $L, \bar{L}$ and $[L, \bar{L}]$ are linearly independent so that

$$\pi_*(L)=\frac{1}{2}(1+iF_u)\frac{\partial}{\partial x}-i\left(x+\frac{1}{2}F_x\right)\frac{\partial}{\partial u}$$

and

$$\pi(M)=\{(x,\ u+iv)\,|\,v=x^2+F(x,0,u)\}$$

where π is the map from M to $\pi(M)$ with

$$\pi(x+iy,\ u+iv(x,y,u))=(x,u+iv(x,0,u)).$$

We further assume that $F(x,y,u)$ is the function of x and u alone.

THEOREM 3.1. *Under the above assumptions there exist a C^∞ function f vanishing to infinite order at the origin and a real vector field Q on $\pi(M)$ such that if v is a C^1 solution to the equation*

$$(\pi_*(L)+fQ)v=0,$$

then $dv=0$ at the origin.

Proof. Take a sequence of curves C_i on $\pi(M)$ converging to the origin such that for each curve we get S_i foliated by the set $\Gamma(\lambda)$. In particular, each C_i is chosen so as not to include any points on Γ_s. Let T_i be the (open) region bounded by S_i.

Let f be a nonnegative C^∞ function whose support is the closure of $\bigcup T_i$. We note that f and all its derivatives are zero at the origin.

Now assume that

$$\pi_*(L)h+fh_u=0$$

in some neighborhood Ω of the origin. By omitting some of the first C_i we may assume that Ω contains $\bigcup T_i$. Then by the theorems 2.2 and 2.3 we have

$$\iint_{T_i}\pi_*(L)h\ dxdu=0$$

and hence

$$\iint_{T_i}fh_u dxdu=0.$$

Therefore in each T_i there must be points p_i and q_i for which Re $h_u(p_i)=0$ and Im $h_u(q_i)=0$. Since $\{p_i\}$ and $\{q_i\}$ converge to the origin, we see that $h_u=0$ at the origin.

Then from the equation $\pi_*(L)h+fh_u=0$ we have $h_x=0$ at the origin.

Thus $dh=0$ at the origin. Q.E.D.

Once the theorem 3.1 is proved, we can follow the same arguments in Nirenberg [4], Treves [5] or Jacobowitz-Treves [2] to get the following.

THEOREM 3.2. *Under the same assumptions as in the theorem* 3.1, *there exist a* C^∞ *function* $\tilde{f}$ *vanishing to infinite order at the origin and a real vector field* $\tilde{Q}$ *on* $\pi(M)$ *such that if* v *is a* C^1 *solution of*

$$[\pi_*(L)+\tilde{f}\tilde{Q}]v=0,$$

then v *is constant in a neighborhood of the origin.*

References

1. Jacobowitz H. and Treves F., *Nonrealizable CR structures,* Invent. Math. 66, 231-249 (1982).
2. Jacobowitz H. and Treves F., *Aberrant CR structures,* Hokkaido Math. J. 12, 276-292 (1983).
3. Kim J. and Kim D., *Local constancy principle and its applications to linear partial differential equations.* J. Korean Math. Soc. 17 (1) 99-112 (1980).
4. Nirenberg L., *Lectures on linear partial differential equations.* Reg. Conf. Ser. in Math. 17, AMS, Providence (1972).
5. Treves F., *Remarks about certain linear PDE's in two variables,* Comm. in P.D.E. 5, 381-425 (1980).

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