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CLASSIFICATION OF IRREDUCIBLE PLANE CURVE SINGULARITIES

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1. Introduction

Let $V=\{(y,z)\mid f(y,z)=0\}$ be an analytic subvariety of a polydisc in C^2 with $(0,0)\in V$ and $(0,0)$ the only singular point where f is holomorphic near $(0,0)$ and square-free. Here we want to consider plane curve singularities in terms of Weierstrass polynomials. Suppose that f is irreducible in $C\langle y,z\rangle$, the ring of convergent power series centered at $(y,z)=(0,0)$. Let $\pi: M\to C^2$ be the composition of a finite number of quadratic transformations such that $\pi: \overline{\pi^{-1}(V-\{(0,0)\})}\to V$ is a resolution of the singular point $(0,0)\in V$. Let $\pi^{-1}(0,0)=E=\bigcup E_i$, $1\leq i\leq m$, be the decomposition of E into irreducible components. Let $V^{(m)}$ be the proper transform of V under π and let $(f\circ\pi)=V^{(m)}+\Sigma e_iE_i$, $1\leq i\leq m$, be the divisor of $f\circ\pi$. Using additional blow-ups we may assume that any two components of $V^{(m)}$ and E meet transversely whenever they meet and no three distinct components of them meet. If there is another analytic subvariety W with its only singular point at $(0,0)$ defined by an irreducible Weierstrass polynomial g, then we define that V and W have a *homeomorphic resolution* if they have equivalent divisors with normal crossings. By this definition we can classify irreducible plane curve singularities in terms of Weierstrass polynomials near the origin. This classification is more understandable. Next, using this result we are going to prove Zariski's classification [Z2] of irreducible curve singularities in terms of parametrization of plane curves. In fact we shall prove that these two classification are equivalent.

2. Known preliminaries

Let $V=\{(y,z)\mid f(y,z)=0\}$ be an analytic subvariety of a polydisc in

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C^2 with $(0,0)\in V$ and $(0,0)$ a singular point where f is holomorphic near $(0,0)$ and square-free. Let $\pi : M\to C^2$ be a quadratic transformation of C^2 at $(0,0)$. Let (u_1, v_1) and (u_2, v_2) be coordinate patches for M with $\pi(u_1, v_1)=(y, z)=(u_1v_1, v_1)$ and $\pi(u_2, v_2)=(y, z)=(v_2, u_2v_2)$. Let e be the multiplicity or total order of f at $(0,0)$. $e\geqq 2$. Then $\pi^{-1}(V)$, the total transform of V is given locally by $f(u_1v_1, v_1)=v_1^e f_1(u_1, v_1)$ and $f(v_2, u_2v_2)=v_2^e f_2(u_2, v_2)$. Let $V^{(1)}$ be defined by $\{f_1(u_1, v_1)=0$ or $f_2(u_2, v_2)=0\}$. Then we call $V^{(1)}$ *the proper transform* of V at $(0,0)$. A quadratic transformation defined as above is often called *a monoidal transformation, a σ-process,* or *a blow-up.* After m iterations of blow-ups, let $\tau_m=\pi\circ\pi_2\circ\cdots\circ\pi_m : M^{(m)}\to C^2$. Let $V^{(m)}$ be the proper transform of V under τ_m. Let $E^{(m)}=\tau_m^{-1}(0,0)$. Then $E^{(m)}$ is, by definition, an exceptional set of the first kind. Let $E^{(m)}=\bigcup E_i^{(m)}$, $1\leq i\leq m$, be the decomposition of $E^{(m)}$ into irreducible components. Let $(f\circ\tau_m)=V^{(m)}+\Sigma e_iE_i^{(m)}$, $1\leq i\leq m$, be the divisor of $f\circ\tau_m$. Then we have the following well-known theorem.

THEOREM 2.1. *Let $V=\{(y,z)\,|\,f(y,z)=0\}$ be an analytic subvariety of a polydisc in C^2 with $(0,0)\in V$ and $(0,0)$ only a singular point where f is holomorphic near $(0,0)$ and square-free. There exists an analytic manifold M by finitely m-times quadratic transformations, $\tau_m : M\to C^2$, such that if R is the set of regular points on V, $\tau_m : \overline{\tau_m^{-1}(R)}\to V$ is a resolution of the singular point $(0,0)$ of V where $\overline{\tau_m^{-1}(R)}$ is the closure of $\tau_m^{-1}(R)$ in M.*

COROLLARY 2.2. *Under the same assumption of Theorem 2.1, after additional quadratic transformations any two components of $V^{(m)}$ and $\bigcup E_i^{(m)}$ meet with normal crossings whenever they meet and no three distinct components of $V^{(m)}$ and $\bigcup E_i^{(m)}$ meet where $V^{(m)}$ and $\bigcup E_i^{(m)}$ are defined just before Theorem 2.1.*

DEFINITION 2.3. Let $V=\{f(y,z)=0\}$ and $W=\{g(y,z)=0\}$ be analytic subvarieties of a polydisc in C^2 with $(0,0)\in V$, $(0,0)\in W$ and $(0,0)$ only singular points where f, g are holomorphic near $(0,0)$ and square-free. V and W are said to *have a homeomorphic resolution* or *to be equisingular* if using the composition of the same number of successive quadratic transformations, $\tau_m : M\to C^2$, $(f\circ\tau_m)$ and $(g\circ\tau_m)$ are equivalent as divisors.

3. A classification of irreducible plane curves in terms of Weierstrass polynomials

Instead of parametrization of plane curves we want to consider plane curve singularities in terms of Weierstrass polynomials. Suppose that f is irreducible in $C\langle y, z\rangle$, the ring of convergent power series centered at $(y, z)=(0, 0)$. Let $f=f(y, z)=z^n+a_1y^{\alpha_1}z^{n-1}+\cdots+a_iy^{\alpha_i}z^{n-i}+\cdots+a_ny^{\alpha_n}$ be irreducible in $C\langle y, z\rangle$ where each $a_i=a_i(y)$, $1\leq i\leq n$, is, if not identically zero, nonvanishing holomorphic near $y=0$ and the α_i are positive integers. Moreover, we may assume that the total order or multiplicity of f at $(0, 0)$ is $n\geqq 2$. Then by [K] we can classify irreducible plane curve singularities at $(0, 0)$ in terms of Weierstrass polynomials as follows (Here $(\beta_1, \cdots, \beta_k)$ is a greatest common divisor of $\beta_1, \cdots, \beta_k$).

The type[1] : $f_1=z^n+y^m$ where $2\leq n<m$ and $(n, m)=1$. If $g_1=z^a+y^b$ is another irreducible polynomial with $a<b$ then f_1 and g_1 have a homemorphic resolution if and only if $a=n$, $b=m$ and $(n, m)=1$. Here is one characteristic of irreducible plane curves belonging to the type [1]. In the sense of Corollary 2. 2, the last exceptional curve only has three distinct intersection points with the other two exceptional curves and the proper transform.

The type[2] : $f_2=(z^a+y^b)^c+y^dz^e$ where $a<b$, $ad+be>abc$, $(a, b)=1$ and $(c, ad+be)=1$. Note that the multiplicity of f_2 is ac. In fact to get a necessary and sufficient condition in the type [2] by use of blow-ups the argument is basically as same as that to get a greatest common divisor of given two integers n and m which can be expressed as a linear combination of n and m, that is seen in the proof of the Euclidean algorithm in the principal ideal domain. If $g_2=(z^A+y^B)^C+y^Dz^E$ is another irreducible polynomial with its multiplicity AC then $a=A$, $b=B$ and $c=C$ and $ad+be=AD+BE$ is a necessary and sufficient condition that f_2 and g_2 have a homeomorphic resolution. Observe that as we have seen in the type [1] there are exactly two exceptional curves each of which has three distinct intersection points. One of these two has an intersection point with the proper transform. Also after a finite number of quadratic transformations the total transform may be defined by $\beta^{abc}((\alpha+1)^c+\beta^{ad+be-abc})$. Note that if $ad+be-abc=1$, then $\beta=0$ is tangent to the proper transform defined by $(\alpha+1)^c+\beta^{ad+be-abc}=0$. Anyway we may assume that the proper transform belongs to the type [1] because we need additional blow-ups to get the result in the sense of Corollary 2. 2.

Finally we may write $f_2=f_1^c+y^dz^e$ where f_1 is the one in the type [1].

The type [3] : $f_3=((z^a+y^b)^c+y^dz^e)^h+(z^a+y^b)^iy^jz^l=f_2^h+f_1^iy^jz^l=(f_1^c+y^dz^e)^h+f_1^iy^jz^l$ where $f_1=z^a+y^b$ and $f_2=(z^a+y^b)^c+y^dz^e$. From the type [1] and the type [2] we can find a necessary condition $(a, b)=1$ and $(c,\ ad+be-abc)=(c,\ ad+be)=1$. To get a necessary and sufficient condition for the type [3], we need a suitable number of blow-ups to get the proper transfrom which may be assumed to belong to the type [2]. Then at that time the total transform is given by $f_3=\beta^{abch}\{[(\alpha+1)^c+\beta^{ad+be-abc}]^h+(\alpha+1)^i\beta^{abi+bl}\times\beta^{aj-abch}\}$. Thus we get another condition that $i(ad+be-abc)+c(abi+bl+aj-abch)$ and h are relatively prime. Summarizing this condition, we get that $i(ad+be)+c(bl+aj)$ and h are relatively prime. Thus the necessary and sufficient condition is that $(a, b)=1$, $(c,\ ad+bc)=1$, $(iad+ibe+cbl+caj,\ h)=1$, $ad+be-abc>0$ and $i(ad+be)+c(bl+aj)-ch(ad+be)>0$. Note that there are exactly three exceptional curves each of which has three distinct intersection points in the sense of Corollary 2. 2.

Now if the multiplicity of f is fixed then inductively we can find the type [k].

The type [k] : $f_k=f_{k-1}^{n_{k-1}}+f_{k-2}^{n_{k-2}}f_{k-3}^{n_{k-3}}\cdots f_1^{n_1}y^{m_1}z^{m_2}$ with some reasonable conditions on $f_1, f_2, \cdots, f_{k-1}$ and integers $n_1, \cdots, n_{k-1}, m_1, m_2$. Observe that there are exactly k distinct exceptional curves each of which has three distinct intersection points in the sense of Corollary 2. 2.

4. A classification of irreducible plane curves in terms of topological parametrization.

In this section we are going to prove Zariski's theorem.

THEOREM (*Zariski*). *Given a germ of reduced and irreducible plane curves* $(X_0, 0)$, *there is a sequence of integers, the characteristic of the curve* $(n, \beta_1, \cdots, \beta_g)=C(X_0, 0)$, *$n$ is the multiplicity of X_0 at 0 and the β_i are the characteristic exponents of the curve such that this character can be described by saying that* $(X_0, 0)$ *has the same topological type as the curve given parametrically by*

$$y=t^n$$
$$z=t^{\beta_1}+t^{\beta_2}+\cdots+t^{\beta_g}$$

where $n<\beta_1<\cdots<\beta_g$ *and* $n>(n_1, \beta_1)>\cdots>(n, \beta_1, \cdots, \beta_g)=1$. *If another* *irreducible plane curve* $(X_1, 0)$ *is defined by* $y=t^m$ *and* $z=t^{\gamma_1}+t^{\gamma_2}+\cdots+t^{\gamma_k}$ *where* $m<\gamma_1<\cdots<\gamma_k$ *and* $m>(m,\gamma_1)>\cdots>(m,\gamma_1,\cdots,\gamma_k)=1$ *then* $(X_1, 0)$ *and* $(X_0, 0)$ *have the same topological type if and only if* $n=m$, $g=h$ *and* $\beta_i=\gamma_i$.

To prove the above the theorem it is just enough to follow Proposition 4.2 and Theorem 4.3.

Let $f(y,z)=a_0z^n+a_1y^{\alpha_1}z^{n-1}+\cdots+a_iy^{\alpha_i}z^{n-i}+\cdots+a_ny^{\alpha_n}$ be irreducible in $C\langle y,z\rangle$ where a_0 and a_n are units in $C\langle y,z\rangle$, each a_i, $1\leq i\leq n-1$, is, if not identically zero, a unit in $C\langle y,z\rangle$ and the α_i are positive integers. By the Weierstrass Preparation Theorem we may write f in the form $f=A(z^n+b_1y^{\beta_1}z^{n-1}+\cdots+b_ny^{\beta_n})$ where A is a unit in $C\langle y,z\rangle$, $b_n=b_n(y)$ is nonvanishing holomorphic near $y=0$, the b_i for $1\leq i\leq n-1$ are nonvanishing holomorphic near $y=0$ if exist and the β_i are positive integers. Note that $\beta_n=\alpha_n$. Next by a nonsingular change of coordinate $(y,z)\to(y, w)=(y, z-(b_1/n)y^{\beta}{}_1)$ we can eliminate the coefficient of z^{n-1} in f without any effect on irreducibility of f in $C\langle y,z\rangle$. Thus we may rewrite f as follows:

$$f=A(w^n+c_2y^{\gamma_2}w^{n-2}+\cdots+c_ny^{\gamma_n})$$

where c_n is nonvanishing holomorphic near $y=0$, the c_i are nonvanishing holomorphic near $y=0$ for $1\leq i\leq n-1$ if exist and the γ_i are positive integers. Then we need to find the relationship between $\alpha_n=\beta_n$ and γ_n.

LEMMA 4.1. *Under the previous discussion we have two cases for the relationship between* $\alpha_n=\beta_n$ *and* γ_n.

(i) *If* $\alpha_n=\beta_n=nk$ *for some positive integer* k *then* $\gamma_n>\beta_n=\alpha_n$ *and* γ_n *is not a multiple of* n.

(ii) *If* $\alpha_n=\beta_n$ *is not a multiple of* n *then* $\alpha_n=\beta_n=\gamma_n$.

Proof. The proof of (i) just follows from Hensel's Lemma after replacing w by cy^kw for some constant c and a positive integer k.

To prove (ii), $\frac{\beta_i}{i}\geqq\frac{\beta_n}{n}$ by [K, Theorem 2.2] and in particular $\beta_1>\frac{\beta_n}{n}$ because β_n/n is not an integer. From $f(y,w)=A[(z-(b_1/n)\cdot y^{\beta_1})^n+\cdots+b_iy^{\beta_i}(z-(b_1/n)\cdot y^{\beta_1})^{n-i}+\cdots+b_ny^{\beta_n}]$ consider monomials containing the variable y only in the bracket. Computing the degree of y in each monomial then

$$\beta_i+(n-i)\beta_1>\frac{i}{n}\beta_n+\frac{n-i}{n}\beta_n=\beta_n \text{ for } i\neq n.$$

Thus we get $\alpha_n=\beta_n=\gamma_n$.

PROPOSITION 4. 2. *Suppose that an analytic subvariety* $V=\{f(y,z)=0\}$ *is locally near* $(y,z)=(0,0)$ *described by*

$$y=t^n$$
$$z=c_1t^{\beta_1}+c_2t^{\beta_2}+\cdots+c_kt^{\beta_k}+\text{terms of degree}>\beta_k$$

where $n<\beta_1<\cdots<\beta_k$, $n>(n,\beta_1)>\cdots>(n,\beta_1,\cdots,\beta_k)=1$ *and the* c_i *are all nonzero coefficients for* $1\leq i\leq k$. *Then* V *is irreducible near* $(y,z)=(0,0)$.

If another analytic subvariety W *is locally near* $(y,z)=(0,0)$ *parametrized by* $y=t^m$ *and* $z=b_1t^{\gamma_1}+b_2t^{\gamma_2}+\cdots+b_ht^{\gamma_h}+$ *terms of degree*$>\gamma_h$ *where* $m<\gamma_1<\cdots<\gamma_h$ *and* $m>(m,\gamma_1)>\cdots>(m,\gamma_1,\cdots,\gamma_h)=1$ *with all nonzero coefficients* b_i *for* $1\leq i\leq h$, *then* V *and* W *have a homeomorphic resolution if and only if* $n=m$, $k=h$ *ane* $\beta_i=\gamma_i$.

Proof. After q iterations of blow-ups which is necessary to get the first exceptional curve which has three distinct intersection points, the proper transform $V^{(q)}$ of V is described by the (u,v)-coordinate for the q-th blow-up below:

$$v=t^{(n,\beta_1)}$$
$$u=c_1'+c_2't^{\beta_2-\beta_1}+\cdots+c_k't^{\beta_k-\beta_1}+\text{terms of degree}>\beta_k-\beta_1$$

for all nonzero coefficients $c_1',\cdots,c_k'$ where we assume that the q-th exceptional curve mentioned above is described by $v=0$. Note that $V^{(q)}$ meets only the q-th exceptional curve. Then by induction on the multiplicity of $V^{(q)}$ and a nonsingular change of coordinate $t\rightarrow t'$ if necessary, we can prove that $V^{(q)}$ is irreducible.

The other part of the above assertion is proved trivially noting that the integer k of β_k in the parametrical expression for the irreducible analytic subvariety V is the number of exceptional curves each of which meets exactly three of other exceptional curves and the proper transform with normal crossings in a sense of Corollary 2. 2.

THEOREM 4. 3. *Let* $V=\{(y,z):f(y,z)=0\}$ *be an analytic subvariety of a polydisc in* C^2 *with* $(0,0)\in V$ *and* $(0,0)$ *the only singular point of* V *where* f *is holomorphic near* $(0,0)$ *and square-free. Assume that* $f=$

$f(y, z)=z^n+b_2y^{\alpha_2}z^{n-2}+\cdots+b_ny^{\alpha_n}$ is irreducible in $C\langle y, z\rangle$ where n is the multiplicity of f, $b_n=b_n(y)$ is nowhere vanishing holomorphic near $y=0$, the $b_i=b_i(y)$ are, if not identically zero, nonvanishing holomorphic near $y=0$ for $2\leq i\leq n-1$ with the coefficient b_1 of z^{n-1} zero and the α_i are positive integers. Observe that α_n is not a multiple of n by Lemma 4.1. Then V can be described parametrically and homeomorphically by

$$y=t^n$$
$$z=t^{\beta_1}+t^{\beta_2}+\cdots+t^{\beta_g} \text{ with } \beta_1=\alpha_n$$

where $n<\beta_1<\beta_2<\cdots<\beta_g$ and $n>(n, \beta_1)>\cdots>(n, \beta_1, \cdots, \beta_g)=1$.

Proof. We are going to prove the theorem by induction on the multiplicity of the above f without coefficient of z^{n-1}. After m iterations of blow-ups which is necessary to get the first appearing exceptional curve which intersect three distinct intersection points, let $\tau_m=\pi\circ\pi_2\circ\cdots\circ\pi_m$: $M\to C^2$ be the composition of such blow-ups. Let $V^{(m)}$ be the proper transform of V under τ_m and let $E^{(m)}=\bigcup E_i$, $1\leq i\leq m$, be the decomposition of $E^{(m)}$ into irreducible components. Let $(f\circ\tau_m)=V^{(m)}+\Sigma e_iE_i$, $1\leq i\leq m$, be the divisor of $f\circ\tau_m$. Let E_m be the m-th exceptional curve. Note that e_m, the coefficient of E_m in $(f\circ\tau_m)$ is the least common multiple of n and α_n. By one of the coordinate patches (u, v) for the blow-up π_m the total transform is described by

$$v^{e_m}h=v^{e_m}[c_0(u+a)^d+\cdots+c_iv^{\gamma_i}(u+a)^{d-i}+\cdots+c_dv^{\gamma_k}]$$

where c_0 and c_d are units in $C\langle u+a, v\rangle$ for a nonzero constant a, the c_i are, if not identically zero, units in $C\langle u+a, v\rangle$ for $1\leq i\leq d-1$ and the γ_i are positive integers. Note that $v=0$ is a local defining equation for E_m, $u=0$ is a local defining equation for the other exceptional curve and that $h=0$, the local defining equation for $V^{(m)}$ meets only E_m. Let $r=\gamma_k$ for brevity. Rewrite $h=A[v^r+\cdots+a_i(u+a)^{\delta_i}v^{r-i}+\cdots+a_r(u+a)^{\delta_r}]$ by the Weierstrass preparation Theorem where A is a unit in $C\langle u+a, v\rangle$, the a_i are nowhere vanishing holomorphic near $u+a=0$ if exist and the δ_i are positive integers. Note that $d=\delta_r$. Then we have three cases: (i) d is a multiple of r, (ii) $d>r$ and d is not a multiple of r and (iii) $r>d$.

(i) d is a multiple of r : Since $d=\delta_r$ is a multiple of r, by eliminating the coefficientc a_1 of v^{r-1} in the expression of h h can be written of the form:

$$A[w^r+b_2(u+a)^{m_2}w^{r-2}+\cdots+b_r(u+a)^{m_r}]$$

where $w=v-(a_1/r)\cdot(u+a)^{\delta_1}$, the b_i are nowhere vanishing holomorphic near $u+a=0$ if exist and the m_i are positive integers. Observe that $m_r>d$ and m_r is not a multiple of r by Lemma 4.1. So by induction assumption h itself can be parametrized homemorphically by

$$u+a=t^r$$
$$w=t^{q_1}+t^{q_2}+\cdots+t^{q_g}$$

where $q_1=m_r$, $r<q_1<q_2<\cdots<q_g$ and $r>(r,q_1)>\cdots>(r, q_1,\cdots,q_g)=1$. Considering the intersection number between $V^{(m)}$ and E_m we can define $V^{(m)}$ with (u,v)-coordinate by

$$u+a=t^r$$
$$v=t^d+t^{q_1}+t^{q_2}+\cdots+t^{q_g}$$

where $r<d<q_1=m_r<q_2<\cdots<q_g$ and $r=(r,d)>(r,d,q_1)>\cdots>(r,d,q_1,\cdots,q_g)=1$. Since $V^{(m)}$ may be described homemorphically by $u+a=t^r+t^{r+q_1-d}+t^{r+q_2-d}+\cdots+t^{r+q_g-d}$ and $v=t^d$, V can be parametrized homeomorphically by

$$y=t^n$$
$$z=t^{\alpha_n}+t^{\alpha_n+r}+t^{\alpha_n+r+q_1-d}+\cdots+t^{\alpha_n+r+q_g-d}.$$

Note that $n<\alpha_n<\alpha_n+r<\alpha_n+r+q_1-d<\cdots<\alpha_n+r+q_g-d$ and $n>(n,\alpha_n)=d>(n,\alpha_n,\alpha_n+r)>(n,\alpha_n,\alpha_n+r, \alpha_n+r+q_1-d)>\cdots>(n,\alpha_n,\alpha_n+r,\alpha_n+r+q_1-d,\cdots,\alpha_n+r+q_g-d)=1$. Thus the case (i) is proved.

The other cases (ii) and (iii) can be proved similarly using Lemma 4.1.

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