Comm. Korean Math. Soc. 1 (1986)

SOME ACTIONS ON THE HYPERFINITE II_1-FACTOR

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1. Introduction

In [2], Choda proved that there are two kinds of ergodic actions of $SL(n, \boldsymbol{Z})$ $(n\geqq 3)$ on the hyperfinite II_1-factor R, one of which constructs full II_1-factors with property T and the other gives full II_1-factors without property T. In this paper we shall study some actions of discrete countably infinite groups on R and prove the same result for $Sp(n, \boldsymbol{Z})$ as the result in [2] for $SL(n, \boldsymbol{Z})$.

2. Preliminaries and Notations

For a fixed positive integer n, define the *symplectic form* $B : \boldsymbol{R}^{2n}\times \boldsymbol{R}^{2n}\to \boldsymbol{R}$ by

$$B(a, b)=\sum_{i=1}^{n} a_i b_{n+i}-\sum_{i=1}^{n} a_{n+i} b_i,$$

where $\boldsymbol{R}$ denotes the set of all real numbers. The symplectic group $Sp(n, \boldsymbol{R})$ is the following subgroup of invertible matrices:

$$\{T\in M_{2n}(\boldsymbol{R}) : B(a, b)=B(Ta, Tb) \text{ for all } a, b\in \boldsymbol{R}^{2n}\}.$$

Denote by K the *semidirect product* $Sp(n, \boldsymbol{R})\times_s \boldsymbol{R}^{2n}$, where multiplication is defined by

$$(S, a)(T, b)=(ST, T^{-1}(a)+b)\ (S, T\in Sp(n, \boldsymbol{R}),\ a, b\in \boldsymbol{R}^{2n}).$$

Let s be an irrational number in $[0, \frac{1}{2}]$ mod. 2π. Put

$$\mu_s((S, a),\ (T, b))=\exp(is\pi B(T^{-1}a, b)),$$

for $S, T\in Sp(n, \boldsymbol{R})$ and $a, b\in \boldsymbol{R}^{2n}$. Then μ_s is a normalized 2-cocycle of $K\times K$ to the torus $\boldsymbol{T}$ (See [3]).

Let's consider the semidirect product $Sp(n, \boldsymbol{Z}) \times_s \boldsymbol{Z}^{2n}$, which is a subgroup of K. Define the left μ_s-regular representation λ^s by

$$(\lambda^s(g)\xi)(h) = \mu_s(h^{-1}, g)\xi(g^{-1}h)$$

for $g, h \in Sp(n, \boldsymbol{Z}) \times_s \boldsymbol{Z}^{2n}$ and $\xi \in l^2(Sp(n, \boldsymbol{Z}) \times_s \boldsymbol{Z}^{2n})$. Then λ^s is a unitary representation of $Sp(n, \boldsymbol{Z}) \times \boldsymbol{Z}^{2n}$ with the cocycle μ_s, that is,

$$\lambda^s(g)\lambda^s(h) = \mu_s(g, h)\lambda^s(gh)$$

for $g, h \in Sp(n, \boldsymbol{Z}) \times_s \boldsymbol{Z}^{2n}$.

The von Neumann algebra R_s generated by $\lambda^s(1, \boldsymbol{Z}^{2n})$ is a hyperfinite II_1-factor ([3]). Define an action α^s of $Sp(n, \boldsymbol{Z})$ on the hyperfinite II_1-factor R_s by

$$\alpha^s(T)(x) = \lambda^s(T, 0)x\lambda^s(T, 0)^*$$

for all $T \in Sp(n, \boldsymbol{Z})$ and all $x \in \lambda^s(Sp(n, \boldsymbol{Z}) \times_s \boldsymbol{Z}^{2n})$.

A countable group G with center $Z(G)$ is said to *have property* F if every inner invariant mean m on G satisfies $m(\chi(Z(G))) = 1$. For $n \geqq 2$, the groups $SL(n, \boldsymbol{Z})$, $GL(n, \boldsymbol{Z})$, $Sp(n, \boldsymbol{Z})$ and the free group F_n on n generators have property F ([2, 3]). Let A be a finite von Neumann algebra with a faithful normal normalized trace τ, G a countable discrete group with property F and α a strongly ergodic action of G on (A, τ). If $Z(G)$ is finite set and α is faithful on $Z(G)$, then the crossed product $R(G, A, \alpha)$ is a full II_1-factor ([4]).

3. Main Result

If there exists an injection j of a countably infinite group G of matrices with integer entries into $Sp(n, \boldsymbol{Z})$, then we get an action of G on R_s by the restriction of α^s on $j(G)$. Let G be one of the groups $Sp(n, \boldsymbol{Z})$, $SL(n, \boldsymbol{Z})$, $GL(n, \boldsymbol{Z})$ and the free group F_2 which is represented by a subgroup of $GL(2, \boldsymbol{Z})$. We shall denote $\alpha^s \circ j$ of G on R_s by the same notation α^s. Then the crossed product $R(G, R_s, \alpha_s)$ is a full II_1-factor since the action α^s is strongly ergodic.

A group G is said to *have property* T *of Kazhdan* iff the trivial representation is an isolated point in the set of equivalence classes of irreducible unitary representations of G. Let Γ be a discrete countable group of matrices with integer entries such that for some n there is an injection j of Γ into $Sp(n, \boldsymbol{Z})$ and semidirect product $j(\Gamma) \times_s \boldsymbol{Z}^{2n}$ has property T. Then the crossed product $R(\Gamma, R_s, \alpha_s)$ is a II_1-factor with property T since the crossed product of a II_1-factor by a countable group of outer automorphisms is also II_1-factor.

Thus we have the following:

THEOREM 1. *Let Γ be as above. Then there are two outer actions α and β of Γ on the hyperfinite II_1-factor R such that the crossed product $R(\Gamma, R, \alpha)$ is a II_1-factor with property T and that $R(\Gamma, R, \beta)$ is a II_1-factor without property T.*

Proof. Let's consider the action α^s of Γ on R_s discussed above. Then $R(\Gamma, R_s, \alpha^s)$ is a II_1-factor with property T. Let β be the Bernoulli shift action of Γ. Then the inner automorphism group is not open ([2]). If a II_1-factor has property T, then the inner automorphism group is open. Hence $R(\Gamma, R, \beta)$ does not have property T.

LEMMA 2 ([3]). *The following groups have property T;*

$$Sp(n, \mathbf{R}) \times_s \mathbf{R}^{2n},\quad Sp(n, \mathbf{Z}) \times_s \mathbf{Z}^{2n} \ \text{for}\ n \geqq 2$$

and

$$j(SL(n, \mathbf{R})) \times_s \mathbf{R}^{2n},\quad j(SL(n, \mathbf{Z})) \times_s \mathbf{Z}^{2n} \ \text{for}\ n \geqq 3.$$

COROLLARY 3. *Let G be one of the groups $Sp(n, \mathbf{Z})$ $(n \geqq 2)$, $SL(n, \mathbf{Z})$ $(n \geqq 3)$. Then there are two kinds of strongly ergodic outer actions α and β of G on R such that $R(G, R, \alpha)$ is a full II_1-factor with property T and that $R(G, R, \beta)$ is a full II_1-factor without property T.*

Proof. Let α^s be the action of G on R_s, then $R(G, R_s, \alpha^s)$ is a full II_1-factor. The Bernoulli shift action β is strongly ergodic and G has property F with finite center. Hence $R(G, R, \beta)$ is a full II_1-factor without property T.

References

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