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ON THE WAVE FRONT SETS OF GENERALIZED DISTRIBUTIONS

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1. Introduction

The wave front set of a distribution is a refinement of the notion of the singular support. It may be thought of as giving at each point the directions in which the distribution is singular. This concept was originally defined by Hörmander [3] and Sato [7], independently. Later Hörmander [4] extended this concept to the ultradistributions defined by Roumieu [6]. (See also in [2]).

In this paper we shall extend this concept to the *generalized distributions* defined by Beurling [1] (see also in [2]), and prove some related results.

For completeness we collect some basic spaces and results which we need in this paper. We denote by $\mathfrak{M}_c$ the set of all continuous real valued functions ω on $\boldsymbol{R}^n$ satisfying the following conditions;

(α) $0=\omega(0)\leq\omega(\xi+\eta)\leq\omega(\xi)+\omega(\eta)$ for all ξ,η in $\boldsymbol{R}^n$.

(β) $\int_{\boldsymbol{R}^n}\frac{\omega(\xi)}{(1+|\xi|)^{n+1}}d\xi<\infty$.

(γ) $\omega(\xi)\geq a+b\log(1+|\xi|)$ for some constants a and $b>0$.

(δ) $\omega(\xi)=\Omega(|\xi|)$ for some concave function Ω on $[0,\infty)$.

Through this paper ω represents an element in $\mathfrak{M}_c$ and Ω is an open set in $\boldsymbol{R}^n$.

We denote by $\mathcal{D}_\omega(\Omega)$ the set of all ϕ in $L^1(\boldsymbol{R}^n)$ such that ϕ has compact support in Ω and

$$\|\phi\|_\lambda=\int_{\boldsymbol{R}^n}|\hat{\phi}(\xi)|e^{\lambda\omega(\xi)}d\xi<\infty$$

for all $\lambda>0$, and $\mathcal{E}_\omega(\Omega)$ the set of all complex valued function ψ in Ω

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such that $\phi\psi\in\mathcal{D}_\omega(\Omega)$ for all $\phi\in\mathcal{D}_\omega(\Omega)$. The dual space of $\mathcal{D}_\omega(\Omega)$ is denoted by $\mathcal{D}_\omega'(\Omega)$ whose elements are called generalized distributions on Ω. Also the dual space $\mathcal{E}_\omega'(\Omega)$ of $\mathcal{E}_\omega(\Omega)$ can be identified with the set of all elements of $\mathcal{D}_\omega'(\Omega)$ which have compact support contained in Ω. For all details reader can see in [2]. The following Paley-Wiener Theorem for generalized distributions will be frequently used in the sequel.

THEOREM A. (Paley-Wiener) *Let K be a compact convex set in $\boldsymbol{R}^n$ with support function H and F an entire function of n complex variables $\zeta=(\zeta_1,\cdots,\zeta_n)=\xi+i\eta$. Then we have*

(i) $F(\zeta)=\int_{R^n} e^{-i\langle x,\zeta\rangle}\phi(x)dx$ *for some* $\phi\in\mathcal{D}_\omega(\boldsymbol{R}^n)$ *with supp* $\phi\subset K$

if and only if for each $\lambda>0$ and each $\varepsilon>0$ there is a constant $C_{\lambda,\varepsilon}$ such that

$$|F(\xi+i\eta)|\leq C_{\lambda,\varepsilon}\exp(H(\eta)+\varepsilon|\eta|-\lambda\omega(\xi))$$

for all $\zeta=\xi+i\eta$ in $\boldsymbol{C}^n$.

(ii) *F is the Fourier-Laplace transform of some $u\in\mathcal{E}_\omega'$ with supp $u\subset K$ if and only if for some $\lambda>0$ and each $\varepsilon>0$ there is a constant C_ε such that*

$$|F(\xi+i\eta)|\leq C_\varepsilon\exp(H(\eta)+\varepsilon|\eta|+\lambda\omega(\xi))$$

for all $\zeta=\xi+i\eta$ in $\boldsymbol{C}^n$.

Before presenting the precise definition of the wave front set of a generalized distribution, we give some motivation for this definition.

LEMMA 1.1. *Let u be in $\mathcal{E}_\omega'(\boldsymbol{R}^n)$. Then u is in $\mathcal{D}_\omega(\boldsymbol{R}^n)$ if and only if, for each integer k there exists a constant C_k such that*

$$|\hat{u}(\xi)|\leq C_k e^{-k\omega(\xi)} \text{ for all } \xi\in\boldsymbol{R}^n.$$

Proof. The necessity is obvious from the Paley-Wiener Theorem. For sufficiency it suffices to show that $\phi u\in\mathcal{D}_\omega(\boldsymbol{R}^n)$ for all ϕ in $\mathcal{D}_\omega(\boldsymbol{R}^n)$, that is,

$$\int_{R^n}|\widehat{\phi u}(\xi)|e^{\lambda\omega(\xi)}d\xi<\infty \text{ for each } \lambda>0.$$

To evaluate this integral, observe

$$|\widehat{\phi u}(\xi)|=(2\pi)^{-n}|\hat{\phi}*\hat{u}(\xi)|$$
$$\leq(2\pi)^{-n}\int_{R^n}|\hat{\phi}(\xi-\eta)||\hat{u}(\eta)|d\eta.$$

Applying the Paley-Wiener theorem and the hypothesis for ϕ and u, respectively, we have, for any positive integers k and l,

$$|\hat{\phi}(\xi-\eta)|\leq C_k e^{-k\omega(\xi-\eta)}$$

and

$$|\hat{u}(\eta)|\leq C_l e^{-l\omega(\eta)}$$

Given $\lambda>0$, let us take k, l so large that $(k-\lambda)b\geq n+1$, $l=k+m$ and $mb\geq n+1$ for the constant b in (γ). Then we obtain

$$|\widehat{\phi u}(\xi)|\leq(2\pi)^{-n}\int_{R^n}C_k e^{-k\omega(\xi-\eta)}C_l e^{-l\omega(\eta)}d\eta$$
$$\leq C\int e^{-k(\omega(\xi-\eta)+\omega(\eta))}e^{-m(a+b\log(1+|\eta|))}d\eta$$
$$\leq Ce^{-k\omega(\xi)}\int\frac{d\eta}{(1+|\eta|)^{n+1}}$$
$$\leq Ce^{-k\omega(\xi)}.$$

Here we used the conditions (γ) and (α), and C represents different constant in each step. Therefore, from the fact $(k-\lambda)b\geq n+1$ we get

$$\int|\widehat{\phi u}(\xi)|e^{\lambda\omega(\xi)}d\xi\leq C\int e^{(\lambda-k)\omega(\xi)}d\xi$$
$$\leq C\int\frac{1}{(1+|\xi|)^{(k-\lambda)b}}d\xi<\infty.$$

Q.E.D.

The Fourier-Laplace transform $\widehat{\phi u}(\xi)$ for $u\in\mathcal{D}_\omega'(\Omega)$ and $\phi\in\mathcal{D}_\omega(\Omega)$ can be written as

$$\widehat{\phi u}(\xi)=\langle u,\ e^{-i\langle\xi,\cdot\rangle}\phi\rangle.$$

From the above lemma we have

COROLLARY 1.2. *Let $u\in\mathcal{D}_\omega'(\Omega)$ and U be an open subset of Ω. Then $u|_U\in\mathcal{E}_\omega(U)$ if and only if for each $\phi\in\mathcal{D}_\omega(U)$ and each integer $k\geq 0$, there is a constant $C_{k,\phi}$ such that*

$$|\langle u, e^{-i\langle\xi,\cdot\rangle}\phi\rangle| \leq C_{k,\phi}e^{-k\omega(\xi)}, \quad \xi\in \boldsymbol{R}^n$$

2. The ω-wave front set

In [2] the ω-singular support of $u\in\mathcal{D}_\omega'(\Omega)$ is defined by the complement of the largest open subset U of Ω such that u is in $\mathcal{E}_\omega(U)$. Motivated by the Corollary 1.2, we introduce a refinement of the notion of ω-singular support of a generalized distribution u on Ω.

DEFINITION. The *ω-wave front set*, denoted by $\omega-WF(u)$, of a generalized distribution $u\in\mathcal{D}_\omega'(\Omega)$ is defined to be the complement in $\Omega\times(\boldsymbol{R}^n-\{0\})$ of the set of all points (x_0, ξ_0) in $\Omega\times(\boldsymbol{R}^n-\{0\})$ such that for some neighborhood U of x_0 and some conic neighborhood V of ξ_0 we have, for each $\phi\in\mathcal{D}_\omega(U)$ and each integer $k\geq 0$ there is a constant $C_{k,\phi}$ such that

$$|\langle u, e^{-i\langle\xi,\cdot\rangle}\phi\rangle| \leq C_{k,\phi}e^{-k\omega(\xi)} \text{ for each } \xi\in V.$$

When $\omega(\xi)=\log(1+|\xi|)$, it is clear, from the fact $\mathcal{D}_\omega(\Omega)=\mathcal{D}(\Omega)$, that $\omega-WF(u)=WF(u)$.

We denote by π the projection of $\Omega\times(\boldsymbol{R}^n-\{0\})$ onto Ω. A subset Γ of $\Omega\times(\boldsymbol{R}^n-\{0\})$ is called a *conic set* or a *cone* if its intersection with each fibre of π is a cone in the usual sense; that is, if $(x, \xi)\in\Gamma$ implies $(x, t\xi)\in\Gamma$ for each $t>0$. Note that complements, unions and intersections of conic sets are conic sets.

LEMMA 2.1. *Let Ω be an open subset of $\boldsymbol{R}^n$ and let $u\in\mathcal{D}_\omega'(\Omega)$.*

(a) *$\omega-WF(u)$ is a closed conic set in $\Omega\times(\boldsymbol{R}^n-\{0\})$.*

(b) *If W is an open subset of Ω, then $\omega-WF(u|_W)$ $=\omega-WF(u)\cap\pi^{-1}(W)$.*

(c) sing_ω supp $u=\pi(\omega-WF(u))$.

(d) *If $\psi\in\mathcal{E}_\omega(\Omega)$, then $\omega-WF(\psi u)\subset\omega-WF(u)$.*

(e) *If $\psi\in\mathcal{E}_\omega(\Omega)$ and $1/\psi\in\mathcal{E}_\omega(W_\psi)$, where $W_\psi=\{x\in\Omega\,|\,\psi(x)\neq 0\}$, then*

$$\omega-WF(\psi u)\cap\pi^{-1}(W_\psi)=\omega-WF(u)\cap\pi^{-1}(W_\psi).$$

Proof. (a) The complement of $\omega-WF(u)$ is the union of the open conic sets $U\times V$ provided by the definition. (b) Obvious. (c) If $x_0\notin\text{sing}_\omega$-supp u, then for some neighborhood U of x_0 $u|_U\in\mathcal{E}_\omega(U)$. So, by Corollary 1.2, for each $\phi\in\mathcal{D}_\omega(U)$ and each integer $k\geq 0$, there is a constant $C_{k,\phi}$ such that

$$|\widehat{\phi u}(\xi)| \leq C_{k,\phi} e^{-k\omega(\xi)} \text{ for each } \xi \in \boldsymbol{R}^n.$$

Hence, $(x_0, \xi_0) \notin \omega - WF(u)$ for all $\xi_0 \in \boldsymbol{R}^n - \{0\}$, i.e., $x_0 \notin \pi(\omega - WF(u))$. Conversely, suppose $x_0 \notin \pi(\omega - WF(u))$. Then $(x_0, \xi_0) \notin \omega - WF(u)$ for all $\xi_0 \in S^{n-1}$, the unit sphere in $\boldsymbol{R}^n$. Thus, for each $(x_0, \xi_0) \in \{x_0\} \times S^{n-1}$, there is a neighborhood U_{ξ_0} of x_0, and a conic neighborhood V_{ξ_0} of ξ_0 such that for each $\phi \in \mathcal{D}_\omega(U_{\xi_0})$ and each integer $k \geq 0$, there is a constant $C_{k,\xi_0,\phi}$ such that

$$|\widehat{\phi u}(\xi)| \leq C_{k,\xi_0,\phi} e^{-k\omega(\xi)} \text{ for each } \xi \in V_{\xi_0}.$$

Since $\{V_{\xi_0} : \xi_0 \in S^{n-1}\}$ covers S^{n-1}, the compactness of S^{n-1} implies that some finite subcollection $\{V_{\xi_i}\}_{i=1}^m$ covers S^{n-1}. Set $V = \bigcup_{i=1}^m V_{\xi_i}$, and $U = \bigcap_{i=1}^m U_{\xi_i}$. Then $V = \boldsymbol{R}^n - \{0\}$ and U is a neighborhood of x_0. Moreover, for any $\phi \in \mathcal{D}_\omega(U)$ and each integer $k \geq 0$, there is a constant $C_{k,\phi} = \max_i C_{k,\xi_i,\phi}$ such that

$$|\widehat{\phi u}(\xi)| \leq C_{k,\phi} e^{-k\omega(\xi)} \text{ for each } \xi \in V = \boldsymbol{R}^n - (0).$$

Hence by Corollary 1.2, $u|_U \in \mathcal{E}_\omega(U)$. So $x_0 \notin \text{sing}_\omega \text{ supp } u$. (d) If $(x_0, \xi_0) \notin \omega - WF(u)$, then there is a neighborhood U of x_0 and a conic neighborhood V of ξ_0 such that for each $\phi \in \mathcal{D}_\omega(U)$ and integer $k \geq 0$, there is a constant $C_{k,\phi}$ such that

$$|\widehat{\phi u}(\xi)| \leq C_{k,\phi} e^{-k\omega(\xi)} \text{ for each } \xi \in V.$$

But, $\phi \in \mathcal{D}_\omega(U)$ implies $\psi\phi \in \mathcal{D}_\omega(U)$. Hence, there is a constant $C_{k,\psi\phi}$ such that

$$|\widehat{\phi(\psi u)}(\xi)| \leq C_{k,\psi\phi} e^{-k\omega(\xi)} \text{ for each } \xi \in V.$$

Thus $(x_0, \xi_0) \notin \omega - WF(\psi u)$. (e) By (b) it suffices to show that $\omega - WF(\psi u | W_\phi) = \omega - WF(u | W_\phi)$, and we may assume that $\phi(x) \neq 0$ for each $x \in \Omega$. Then (e) follows from (d) since $u = (1/\psi)(\psi u)$ and $1/\psi \in \mathcal{E}_\omega(W_\phi)$.

Q.E.D.

Let $P = \Sigma a_\alpha D^\alpha$, $a_\alpha \in \mathcal{E}_\omega(\Omega)$, and $|\alpha| \leq m$. Then, clearly, we have supp— $Pu \subset \text{supp } u$ for each $u \in \mathcal{D}_\omega'(\Omega)$. This property is called the local property.

We also have sing_ω supp $Pu \subset \text{sing}_\omega$ supp u. This property is a version of the pseudo-local property. Indeed, we have the following stronger version of the pseudo-local property:

THEOREM 2.2. (Strong Pseudo-Local Property). *If* $P=\Sigma a_\alpha D^\alpha$, $a_\alpha \in \mathcal{E}_\omega(\Omega)$, *and* $|\alpha| \leq m$, *then* $\omega-WF(Pu) \subset \omega-WF(u)$ *for each* $u \in \mathcal{D}_\omega'(\Omega)$.

Proof. In view of Lemma 2.1 (d) it suffices to show that $\omega-WF(D_ju) \subset \omega-WF(u)$. If $\phi \in \mathcal{D}_\omega(\Omega)$ then $\phi D_ju = D_j(\phi u) - (D_j\phi)u$. And if $(x_0, \xi_0) \notin \omega-WF(u)$, then there is a neighborhood U of x_0 in Ω and a conic neighborhood V of ξ_0 in $\boldsymbol{R}^n - \{0\}$ such that for each $\phi \in \mathcal{D}_\omega(U)$ and each integer $k \geq 0$, there is a constant $C_{k,\phi}$ such that

$$|\widehat{\phi u}(\xi)| \leq C_{k,\phi} e^{-k\omega(\xi)} \text{ for each } \xi \in V.$$

Then

$$|\widehat{D_j(\phi u)}(\xi)| \leq |\xi_j| C_{k,\phi} e^{-k\omega(\xi)} \text{ for each } \xi \in V,$$

and

$$|\widehat{(D_j\phi) u}(\xi)| \leq C_{k,D_j\phi} e^{-k\omega(\xi)} \text{ for each } \xi \in V,$$

Thus, we have

$$|\widehat{\phi D_j u}(\xi)| \leq |\xi_j| C_{k,\phi} e^{-k\omega(\xi)} + C_{k,D_j\phi} e^{-k\omega(\xi)}$$

for each $\xi \in V$.

In view of the condition (γ) we have the following inequality:

$$|\xi_j| \leq 1 + |\xi| \leq e^{1/b(\omega(\xi)-a)}$$

Applying this result to the above estimation, we have

$$\begin{aligned} |\widehat{\phi D_j u}(\xi)| &\leq C_{k,\phi} e^{-a/b} e^{1/b\omega(\xi)} e^{-k\omega(\xi)} + C_{k,D_j\phi} e^{-k\omega(\xi)} \\ &\leq C'_{k,\phi} e^{-(k-1/b)\omega(\xi)} \text{ for each } \xi \in V, \end{aligned}$$

which shows $(x_0, \xi_0) \notin \omega-WF(D_ju)$. That is, $\omega-WF(D_ju) \subset \omega-WF(u)$. Q.E.D.

In the definition of $\omega-WF(u)$ we have to estimate an inconveniently large number of functions ϕ. The following theorem shows that we can get away with fewer estimates. From now on we denote by W_ϕ the open set $\{x\in\Omega : \phi(x)\neq 0\}$ for $\phi\in\mathcal{E}_\omega(\Omega)$.

THEOREM 2.3. *If $u\in\mathcal{D}_\omega'(\Omega)$, then $\omega-WF(u)$ is the complement in $\Omega\times\boldsymbol{R}^n-\{0\}$ of the set of all points $(x_0,\xi_0)\in\Omega\times(\boldsymbol{R}^n-\{0\})$ such that for some $\phi\in\mathcal{D}_\omega(\Omega)$ with $\phi(x_0)\neq 0$ and $1/\phi\in\mathcal{E}_\omega(W_\phi)$, there is a conic neighborhood V of ξ_0 in $\boldsymbol{R}^n-\{0\}$ such that for each integer $k\geq 0$ there is a constant C_k such that*

$$|\langle u,\ e^{-i\langle\xi,\cdot\rangle}\phi\rangle|\leq C_k e^{-k\omega(\xi)} \text{ for each } \xi\in V.$$

Proof. Let $\omega-WF'(u)$ be the set defined by the hypotheses of the theorem. If $(x_0,\xi_0)\notin\omega-WF(u)$, then for each $\phi\in\mathcal{D}_\omega(U)$, we have

$$|\widehat{\phi u}(\xi)|\leq C_{k,\phi}e^{-k\omega(\xi)},\ \text{for } \xi\in V.$$

In particular, for $\phi\in\mathcal{D}_\omega(U)$ with $\phi(x_0)\neq 0$ and $1/\phi\in\mathcal{E}_\omega(W_\phi)$, it is true, so $(x_0,\xi_0)\notin\omega-WF'(u)$. Conversely, suppose $(x_0,\xi_0)\notin\omega-WF'(u)$. Then there is a function $\phi\in\mathcal{D}_\omega(\Omega)$ with $\phi(x_0)\neq 0$ and $1/\phi\in\mathcal{E}_\omega(W_\phi)$, and a conic neighborhood V of ξ_0 in $\boldsymbol{R}^n-\{0\}$ such that for each integer $k\geq 0$, there is a constant C_k such that

$$|\widehat{\phi u}(\xi)|\leq C_k e^{-k\omega(\xi)} \text{ for each } \xi\in V.$$

To show $(x_0,\xi_0)\notin\omega-WF(u)$ it suffices, in view of Lemma 2.1 (e), to prove $(x_0,\xi_0)\notin\omega-WF(\phi u)$. Let $\psi\in\mathcal{D}_\omega(\boldsymbol{R}^n)$. Then

$$\widehat{\psi\phi u}(\xi)=(2\pi)^{-n}\int\hat{\psi}(\xi-\eta)\widehat{\phi u}(\eta)\,d\eta.$$

Let

$$I_1(\xi)=\int_V\hat{\psi}(\xi-\eta)\widehat{\phi u}(\eta)\,d\eta$$
$$I_2(\xi)=\int_{\boldsymbol{R}^n-V}\hat{\psi}(\xi-\eta)\widehat{\phi u}(\eta)\,d\eta.$$

We will estimate these integrals. According to the Paley-Wiener Theorem, for each integer $k\geq 0$, there is a constant A_k such that

$$|\hat{\psi}(\xi)|\leq A_k e^{-k\omega(\xi)} \text{ for each } \xi\in\boldsymbol{R}^n.$$

Hence

$$|I_1(\xi)|\leq C_{k+p}A_k\int_V e^{-k\omega(\xi-\eta)}e^{(-k-p)\omega(\eta)}d\eta.$$
$$\leq C_{k+p}A_k e^{-k\omega(\xi)}\int_V e^{-p\omega(\xi)}d\eta.$$

If we choose the integer p with $pb>n+1$, the integral converges and so for each integer $k\geq 0$ we have a constant C_k' such that

$$|I_1(\xi)|\leq C_k' e^{-k\omega(\xi)} \text{ for each } \xi\in\mathbf{R}^n.$$

We now estimate $I_2(\xi)$. Since $\phi u\in\mathcal{E}_\omega'(\mathbf{R}^n)$, by the Paley-Wiener Theorem there are constants B and M such that

$$|\widehat{\phi u}(\xi)|\leq Be^{M\omega(\xi)} \text{ for all } \xi\in\mathbf{R}^n.$$
$$|I_2(\xi)|\leq A_{2M+k+p}B\int_{\mathbf{R}^n-V} e^{(-2M-k-p)\omega(\xi-\eta)}e^{M\omega(\eta)}d\eta$$
$$\leq A_{2M+k+p}B\int_{\mathbf{R}^n-V} e^{(-M-k-p)\omega(\xi-\eta)}e^{M\omega(\xi)}d\eta$$

where we have used the condition (γ). Now let V' be an open cone in V such that the closure of $V'\cap S^{n-1}$ is compact in V. Let $\delta>0$ be the distance from $V'\cap S^{n-1}$ to the boundary of V, and n_0 be an integer greater than $1/\delta$. If $\xi\in V'$ and $\eta\in\mathbf{R}^n-V$, then $\xi/|\xi|\in V'\cap S^{n-1}$ and $\eta/|\xi|\in\mathbf{R}^n-V$. So $|\xi-\eta|\geq\delta|\xi|\geq(1/n_0)|\xi|$. Thus $\omega(n_0(\xi-\eta))\geq\omega(\xi)$. Taking p so large that $pb>n+1$, it follows, from (γ), that for $\xi\in V'$

$$|I_2(\xi)|\leq A_{2M+k+p}Be^{M\omega(\xi)}\int_{\mathbf{R}^n-V} e^{(-M-k)/n_0\,\omega(\xi)}e^{-p\omega(\xi-\eta)}d\eta$$
$$\leq A_{2M+k+p}Be^{(M-M/n_0-k/n_0)\omega(\xi)}\int_{\mathbf{R}^n-V} e^{-pa}(1+|\xi-\eta|)^{-pb}d\eta$$
$$\leq A_{2M+k+p}BCe^{(M-M/n_0-k/n_0)\omega(\xi)}.$$

Hence, for each integer $k\geq 0$, considering n_0k+n_0M-M, there is a constant C_k'' such that

$$|I_2(\xi)|\leq C_k''e^{-k\omega(\xi)} \text{ for each } \xi\in V'.$$

Choosing $C_{k,\phi}=C_k'+C_k''$, we get

$$|\widehat{\psi\phi u}(\xi)|\leq C_{k,\phi}e^{-k\omega(\xi)} \text{ for each } \xi\in V'.$$

Therefore, $(x_0,\xi_0)\notin\omega-WF(\phi u)$.

Q.E.D.

We remark that, from the fact $\phi^{-1}\in\mathcal{E}_\omega(W_\phi)$ when $\omega(\xi)=\log(1+|\xi|)$ and $\phi^{-1}\in A(W_\phi)$ for an analytic function ϕ, we expect that $\phi^{-1}\in\mathcal{E}_\omega(W_\phi)$ in general.

THEOREM 2.4. *Suppose that there are constants a and $b>0$ such that $a+b\omega_1(\xi)\leq\omega_2(\xi)$ for all $\xi\in\boldsymbol{R}^n$. Then $\omega_1-WF(u)\subset\omega_2-WF(u)$ for $u\in\mathcal{D}_{\omega_1}'(\Omega)$.*

Proof. Let $(x_0,\xi_0)\notin\omega_2-WF(u)$. Then, by Theorem 2.3 there is a function $\phi\in\mathcal{D}_{\omega_2}(\Omega)$ with $\phi(x_0)\neq0$ and $\phi^{-1}\in\mathcal{E}_{\omega_2}(W_\phi)$, and a conic neighborhood V of ξ_0 in $\boldsymbol{R}^n-\{0\}$ such that for each integer $k\geq0$, there exists a constant C_k such that

$$|\widehat{\phi u}(\xi)|\leq C_k e^{-k\omega_2(\xi)} \text{ for each } \xi\in V.$$

Then $\phi\in\mathcal{D}_{\omega_1}(\Omega)$ and $\phi(x_0)\neq0$ and

$$\begin{aligned}|\widehat{\phi u}(\xi)|&\leq C_k e^{-k\omega_2(\xi)}\\&\leq C_k' e^{-kb\omega_1(\xi)} \text{ for each } \xi\in V.\end{aligned}$$

It remains to prove that $\phi^{-1}\in\mathcal{E}_{\omega_2}(W_\phi)\subset\mathcal{E}_{\omega_1}(W_\phi)$. We clearly have $\mathcal{D}_{\omega_2}(\Omega)\subset\mathcal{D}_{\omega_1}(\Omega)$. In [2], G. Björck proves that the definition of $\mathcal{E}_\omega(\Omega)$ is equivalent to the following definition: $\mathcal{E}_\omega(\Omega)$ is the set of all complex-valued functions ϕ in Ω such that for each compact subset K of Ω the restriction to K of ϕ and of some $\psi\in\mathcal{D}_\omega(\Omega)$ agree.

Now for each compact subset K of W_ϕ, $\phi^{-1}|_K=\psi$ for some $\psi\in\mathcal{D}_{\omega_2}(\Omega)$. Since $\psi\in\mathcal{D}_{\omega_1}(\Omega)$, it follows that $\phi^{-1}\in\mathcal{E}_{\omega_1}(\Omega)$.

Q.E.D.

Regarding the cotangent bundle $T^*(\Omega)$ as $\Omega\times\boldsymbol{R}^n$ by means of the canonical coordinates we can consider $\omega-WF(u)$ as a subset of $T^*(\Omega)$. In this case we have:

THEOREM 2.5. *Let Ω be an open subset of $\boldsymbol{R}^n$ and $u\in\mathcal{D}_\omega'(\Omega)$. If $(x_0,\xi_0)\in\Omega\times\boldsymbol{R}^n-\{0\}$, then $(x_0,\xi_0)\notin\omega-WF(u)$ if for each real valued $\psi_0\in\mathcal{E}_\omega(\Omega)$ with $d\psi_0(x_0)=\xi_0$ there is a neighborhood W of ψ_0 in $\mathcal{E}_\omega(\Omega)$ (real-valued) and a function $\phi\in\mathcal{D}_\omega(\Omega)$ with $\phi(x_0)\neq0$ and $\phi^{-1}\in\mathcal{E}_\omega(W_\phi)$ such that for each integer $k\geq0$ and each bounded subset $B\subset W$ we have*

$$\langle u,\ e^{-it\psi}\phi\rangle=O(e^{-k\Omega(t)}) \text{ as } t\to\infty, \text{ uniformly for } \psi\in B,$$

where $\Omega(t)$ is the concave function in (δ).

Proof. Assume the hypotheses of the theorem hold. Let $\psi_0(x)=\langle\xi_0, x\rangle$ so $d\psi_0(x_0)=\xi_0$. By using Theorem 1.5.26 in G. Björck [2] we can easily prove that the map $\boldsymbol{R}^n\to\mathcal{E}_\omega(\Omega)$ given by $\xi\to\langle\xi,\cdot\rangle$ is continuous. Thus, there is a neighborhood V of ξ_0 such that $\xi\in V$ implies $\langle\xi,\cdot\rangle\in W$. Moreover, we may choose V compact so that $\langle\xi,\cdot\rangle$ lies in a compact and therefore bounded subset of W. Then by hypotheses, we have

$$\langle u,\ e^{-it\langle\xi,\cdot\rangle}\phi\rangle=O(e^{-k\Omega(t)})\ \text{ as } t\to\infty,\ \text{ uniformly for } \xi\in V.$$

Hence

$$e^{k\Omega(t)}|\langle u,\ e^{-it\langle\xi,\cdot\rangle}\phi\rangle|\le M' \text{ if } t\ge c,\ \xi\in V \text{ for some constant } c>0.$$

Since the sets $\{t\,|\,0\le t\le c\}$ and V are compact, we also have

$$e^{k\Omega(t)}|\langle u,\ e^{-it\langle\xi,\cdot\rangle}\phi\rangle|\le M \text{ for } t>0,\ \xi\in V, \text{ for some constant } M.$$

Set $V'=\{t\xi\,|\,t>0,\ \xi\in V\}$. Then V' is a conic neighborhood of ξ_0. And we may assume, by choosing V appropriately, that there is a rational constant $t_0=q/p>0$, depending only on V, such that $t_0\xi'/|\xi'|\in V$ for all $\xi'\in V'$. Then we have

$$|\langle u,\ e^{-it\langle t_0\xi'/|\xi'|,\cdot\rangle}\phi\rangle|\le Me^{-k\Omega(t)} \text{ for } t>0,\ \xi'\in V'.$$

By taking $t=|\xi'|/t_0$, we get

$$\begin{aligned}|\widehat{\phi u}(\xi')|&\le Me^{-k\Omega(|\xi'|/t_0)}\\&\le Me^{-k\omega\left(\frac{p}{q}\xi'\right)}\\&\le Me^{-\frac{k}{q}\omega(p\xi')}\\&\le Me^{-\frac{k}{q}\omega(\xi')} \text{ for } \xi'\in V',\end{aligned}$$

where the second inequality follows from (α) and the third one follows from the monotonicity of $\Omega(t)$. Since q is independent of k, for each integer $k\ge0$, there is a constant M' such that

$$|\widehat{\phi u}(\xi)|\le M'e^{-k\omega(\xi)} \text{ for each } \xi\in V'.$$

Therefore, by Theorem 2.4, $(x_0,\xi_0)\notin\omega-WF(u)$.

Q.E.D.

In the case of a generalized distribution with compact support we have a nice characterization of the totality of ω-singular directions. Let

$$\Pi' : \Omega \times (\boldsymbol{R}^n - \{0\}) \rightarrow \boldsymbol{R}^n - \{0\}$$

be the projection map.

THEOREM 2.6. *Let $u \in \mathcal{E}_\omega'(\Omega)$ and let $\Gamma_0 = \Pi'(\omega - WF(u))$. Then Γ_0 is a closed cone in $\boldsymbol{R}^n - \{0\}$. If Γ is a closed cone in $\boldsymbol{R}^n - \{0\}$ and $\Gamma \cap \Gamma_0 = \phi$ then for each integer $k \geq 0$ there is a constant $C_{k,\Gamma}$ such that*

$$|\hat{u}(\xi)| \leq C_{k,\Gamma} e^{-k\omega(\xi)} \quad \text{for each } \xi \in \Gamma. \qquad (*)$$

Conversely, if Γ is a closed cone in $\boldsymbol{R}^n - \{0\}$ such that (*) holds for each integer $k \geq 0$ then Γ_0 is disjoint from the interior of Γ. In particular, Γ_0 is the smallest closed cone in $\boldsymbol{R}^n - \{0\}$ such that (*) holds for each closed cone Γ disjoint from Γ_0.

Proof. Let $K = \text{supp } u$. Then $\omega - WF(u) \cap (\Omega \times S^{n-1}) = \omega - WF(u) \cap (K \times S^{n-1})$ is compact. Its image under Π' is compact, and so the set $\Gamma_0 = \{t\xi \mid t > 0,\ \xi \in \Pi'(\omega - WF(u) \cap (\Omega \times S^{n-1}))\}$ is closed in $\boldsymbol{R}^n - \{0\}$. For the last part of the theorem, assume that Γ_1 is a closed cone such that (*) holds for each closed cone Γ disjoint from Γ_1. If $\xi_0 \notin \Gamma_1$, let Γ be a closed cone disjoint from Γ_1 with $\xi_0 \in \text{Int } \Gamma$. Assuming the first part of the theorem Γ_0 is disjoint from the interior of Γ, that is, $\xi_0 \notin \Gamma_0$. Hence $\Gamma_0 \subset \Gamma_1$.

Consider now the first part of the theorem. We prove the converse part first. Let Γ be a closed cone such that (*) holds for each integer $k \geq 0$. Let ξ_0 be an interior point of Γ and choose an open cone V with $\xi_0 \in V \subset \Gamma$. Let $\phi \in \mathcal{D}_\omega(\Omega)$ be such that $\phi = 1$ in a neighborhood of K and $\phi^{-1} \in \mathcal{E}_\omega(W_\phi)$. Then $\phi u = u$. And then

$$|\widehat{\phi u}(\xi)| = |\hat{u}(\xi)| \leq C_{k,\Gamma} e^{-k\omega(\xi)} \text{ for each } \xi \in V \text{ and } k \geq 0.$$

If $x \in K$ then $\phi(x) = 1$ and hence by Theorem 2.3, $(x, \xi_0) \notin \omega - WF(u)$. Thus $\xi_0 \notin \Pi'(\omega - WF(u)) = \Gamma_0$.

For the direct part, let Γ be a closed cone such that $\Gamma \cap \Gamma_0 = \phi$. Then $\Omega \times \Gamma$ is disjoint from $\omega - WF(u)$. Fix $x \in K$. For each $\xi \in \Gamma \cap S^{n-1}$ there is an open neighborhood U' of x and a conic open neighborhood V of ξ and a constant C_k, depending on ϕ, U' and V, such that

$$|\widehat{\phi u}(\xi)| \leq C_k e^{-k\omega(\xi)} \text{ for each } k \geq 0,\ \phi \in \mathcal{D}_\omega(U') \text{ and } \xi \in V.$$

A finite number of the neighborhoods V cover $\Gamma\cap S^{n-1}$. If we let U be the intersection of the corresponding U's, we obtain an open neighborhood U of x such that

$$|\widehat{\phi u}(\xi)|\leq C_k' e^{-k\omega(\xi)} \text{ for each } k\geq 0,\ \phi\in\mathcal{D}_\omega(U) \text{ and } \xi\in\Gamma.$$

A finite number of the neighborhoods U cover K. If $\{\phi_j\}$ is a finite partition of unity in a neighborhood of K with each ϕ_j having support in one of the neighborhoods U and with $\phi_j\in\mathcal{D}_\omega(\Omega)$, then (*) holds in Γ since $u=\Sigma\phi_j u$.

Q.E.D.

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