Comm. Korean Math. Soc 1 (1986)

ON THE LOWEST EIGENVALUE OF THE LAPLACIAN FOR INTERSECTION OF A BOUNDED DOMAIN AND A DOMAIN VANISHING AT ∞.

DONG PYO CHI, IL HAE LEE AND SANG MOON KIM

1. Introduction

Let A and B be two domains in R^n, $n\geq 1$, $\lambda(A)$ and $\lambda(B)$ be the lowest eigenvalues of Δ with Dirichlet boundary conditions. Let B_x denote B translated by $x\in R^n$. Lieb [5] proved that for any $\varepsilon>0$ there is a set X with positive measure such that

$$\lambda(A)+\lambda(B)+\varepsilon>\lambda(A\cap B_x) \quad (x\in X).$$

Further Lieb [5] proved that if A and B are bounded then there is an open set X such that

$$\lambda(A)+\lambda(B)>\lambda(A\cap B_x) \quad (x\in X).$$

A domain A is said to *be vanishing at* ∞ if

$$\mathrm{Vol}\{y \mid |x-y|<1\}\to 0 \text{ as } x\to\infty.$$

In this paper we prove that if A is bounded and B is vanishing at ∞ then there is a set X of positive measure such that

$$\lambda(A)+\lambda(B)>\lambda(A\cap B_x) \quad (x\in X).$$

Let Ω be an open set in R^n and k a positive integer. For $1\leq p<\infty$, we denote by $W^{k,p}(\Omega)$ the set of all functions $f(x)$ defined on Ω such that f and its distributional derivatives $D^s f$ of f orders $|s|=\sum_{j=1}^{n} s_j\leq k$ all belong to $L^p(\Omega)$. Then $W^{k,p}(\Omega)$ is a normed linear space with the norm defined by

$$\|f\|_{k,p}=(\sum_{|s|\leq k}\int_{\Omega}|D^s f|^p dx)^{1/p}.$$

Hence $W^{k,2}(\Omega)$ is a Hilbert space with the scalar product

$$(f,g)=(\sum_{|s|\leq k}\int_{\Omega}D^s f\cdot D^s g dx)^{1/2}.$$

Let $H_0^k(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in $W^{k,2}(\Omega)$, where $C_0^\infty(\Omega)$ denotes the set of all C^∞ functions supported in Ω.

For an open set A in R^n, we define the lowest eigenvalue $\lambda(A)$ of $\Delta=-\left(\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right)$ as following

$$(*)\quad \lambda(A)=\operatorname{Inf}\{E(f)\,|\,f\in H_0^1(A),\ f\neq 0\}$$
$$=\operatorname{Inf}\{E(f)\,|\,f\in C_0^\infty(A),\ f\neq 0\},$$
$$E(f)=\int_A\sum_{i=1}^{n}\left|\frac{\partial f}{\partial x_i}\right|^2 dx\Big/\int_A|f|^2 dx.$$

The reasion of defining the lowest eigenvalue of Δ as the above is well known but for the convenience of readers we give it in the following. For this we need to define the Green's operator of Δ. We adopt the following Lemma from [3].

LEMMA 1. *Let Ω be a bounded (or vanishing at ∞) domain in R^n and $g\in L^2(\Omega)$ then there is a unique $f\in H_0^1(\Omega)$ such that*

$$(g,\phi)=(f,\ (I+\Delta)\phi)$$

for all $\phi\in C_0^\infty(\Omega)$. The mapping defined by $T(g)=f$ from $L^2(\Omega)$ to $H_0^1(\Omega)$ is bounded and $T: L^2(\Omega)\to L^2(\Omega)$ is compact and selfadjoint.

Proof. For $\phi,\psi\in C_0^\infty(\Omega)$ define the Dirichlet inner product and Dirichlet norm respectively by

$$(\phi,\psi)_d=(\phi,\psi)+(d\phi,\ d\psi)=(\phi,\ (I+\Delta)\psi),$$
$$\|\phi\|_d^2=(\phi,\phi)d=\|\phi\|^2+\|d\phi\|^2.$$

It is clear that the Dirichlet norm is equivalent to the Sobolev 1-norm on $H_0^1(\Omega)$. From the inequality

$$|(g,\phi)|\leq\|g\|\ \ \|\phi\|\leq\|g\|\ \ \|\phi\|_d,$$

we know that the linear functional

(1) $$\Lambda_g : \phi \to (g, \phi) \quad (\phi \in C_0^\infty(\Omega)),$$

extends to a bounded linear form on $H_0^1(\Omega)$ with the Dirichet norm. Hence to the linear form Λ_g corresponds a unique element $f \in H_0^1(\Omega)$ such that

$$(g, \phi) = (f, \phi)_d \quad (\phi \in C_0^\infty(\Omega)).$$

The above equality induces a linear operator $Tg = f$, $T : L^2(\Omega) \to H_0^1(\Omega)$, which is characterized by

(2) $$\begin{aligned}(g, \phi) &= (f, \phi)_d = (Tg, \phi)_d \\ &= (Tg, \ (I+\Delta)\phi) \quad (\phi \in C_0^\infty(\Omega)).\end{aligned}$$

Since I and Δ are self-adjoint, T is also self-adjoint. For $g \in L^2(\Omega)$, we know that

$$\Lambda_g(\phi) = (g, \phi) = (Tg, \ (I+\Delta)\phi) \quad (\phi \in C_0^\infty(\Omega)).$$

For $\psi \in H_0^1(\Omega)$, set

$$(Tg, \ (I+\Delta)\psi) = (Tg, \psi) + (dTg, \ d\psi),$$

then by (2) we have

(3) $$\Lambda_g(\psi) = (g, \psi) = (Tg, \ (I+\Delta)\psi).$$

From (3) we deduce that

(4) $$\begin{aligned}\| Tg \|_1 &\leq (Tg, \ Tg)_d \\ &= (Tg, \ (I+\Delta) Tg) \\ &= (g, \ Tg) \\ &\leq \| g \| \ \| Tg \|.\end{aligned}$$

Now an application of the inequality

$$2ab \leq (e/\varepsilon)^2 a^2 + \varepsilon^2 b^2$$

to (4), gives

$$2 \| Tg \|_1 \leq (e/\varepsilon)^2 \| Tg \| + (\varepsilon)^2 \| g \|.$$

Hence

(5) $$\| Tg \|_1 \leq C \| g \|, \quad C > 0.$$

By (5) we know that T is a bounded linear map from $L^2(\Omega)$ to $H_0^1(\Omega)$. The Rellich's LEMMA says that the map

$$H_0^k(\Omega) \longrightarrow H_0^{k-1}(\Omega)$$

is compact. Hence we know the map

$$T : L^2(\Omega) \to H_0^1(\Omega) \to L^2(\Omega)$$

is compact and self-adjoint. This proves the Lemma.

Now we are ready to define the Green's operator. Since T is compact and self-adjoint there is a Hilbert space decomposition

$$L^2(\Omega) = \bigoplus_m E(\rho_m),$$

where ρ_m denotesthe eigenvalue of T and $E(\rho_m)$ the finite dimmensional eigenspace.

Since T is one to one, we know that $\rho_m \neq 0$.

$$Tg = \rho_m g \quad (g \in H_0^1(\Omega)),$$

is equivalent to

$$(g, \psi) = (\rho_m g, \quad (I + \Delta)\phi) \quad \phi \in C_0^\infty(\Omega).$$

It follows that g is an eigenvalue of T is equivalent to

$$\Delta g = \left(\frac{1 - \rho_m}{\rho_m}\right) g \quad (g \in H_0^1(\Omega)),$$

in the sense of distribution.

Hence T and Δ have the same eigen spaces. Set

$$\lambda_m = (1 - \rho_m) / \rho_m,$$

then $\rho_m \to 0$ and $\lambda_m \to \infty$.

If we assume that $\lambda_m \neq 0$ $(\rho_m \neq 1)$ then we have

(6) $$0 < \lambda_1 < \lambda_2 <, \cdots, < \lambda_n <, \cdots.$$

Define the Green's operator by

$$G(\phi) = \frac{1}{\lambda_m}\phi \quad \phi \in E\left(\frac{1}{1 + \lambda_m}\right),$$

then G is compact and self-adjoint. Hence G has the spectral decomposition

$$L^2(\Omega)=\bigoplus_m E(\rho_m),$$

where

$$G(\phi)=\left(\frac{\rho_m}{1-\rho_m}\right)\phi \quad (\phi\in E(\rho_m)).$$

Let Ω be open in R^n. It is well known that for each $\phi\in C_0^\infty(\Omega)$

(7) $$\Delta\psi=\phi,$$

has a C^∞ solution, see [6]. If we require that $\psi\in C_0^\infty(\Omega)$ then the solution is unique. Hence Δ is a surjection on $C_0^\infty(\Omega)$, one can deduce that G maps $C_0^\infty(\Omega)$ onto $C_0^\infty(\Omega)$. By the spectral decomposition theorem of compact self-adjoint operators, we have for any $u\in C_0^\infty(\Omega)$

(8) $$u=\Sigma b_{m_i}\phi_{m_i}, \quad \phi_{m_i}\in E(1/\lambda_m), \quad \|\phi_{m_i}\|=1.$$

Hence $G^2u\in C_0^\infty(\Omega)$ and

(9) $$G^2u=\Sigma a_{m_i}\phi_{m_i}.$$

By (8) and (9) we have

$$b_{m_i}=a_{m_i}/(\lambda_m)^2.$$

It follows that each $v\in C_0^\infty(\Omega)$ has a representation

(10) $$v=G^2u=\Sigma(a_{m_i}/(\lambda_m)^2)\phi_{m_i},$$

$$u=\sum_{i=1}^{m}a_{m_i}\phi_{m_i}.$$

For $v\in C_0^\infty(\Omega)$, the Green's theorem says that

(11) $$\int(\nabla v,\ \nabla v)dx=\int v\Delta v dx.$$

Since

$$\Delta v=\Sigma\frac{a_{m_i}\lambda_m}{(\lambda_m)^2}\phi_{m_i},$$

using (10) and (11) we deduce that

$$(12)\qquad \frac{\int(\nabla v,\ \nabla v)dx}{\int(v,\ v)dx}=\frac{\int v\ \Delta v\ dx}{\int v^2\ dx}$$
$$=(\Sigma a_{m_i}{}^2\lambda_m/\Sigma a_{m_i}{}^2).$$

In (12) the possible candidate of the minimum value is λ_1. This gives the meaning of definition (*).

For any $\phi\in C_0^\infty(\Omega)$ we know that the equation

$$(I+\Delta)u=\phi$$

has a solution in $C_0^\infty(\Omega)$, see [6]. By this property of $(I+\Delta)$ and the definition of T, one can prove that

$$(13)\qquad dT=Td\ \ (\partial T=T\partial \text{ in the complex case}),$$

and

$$(14)\qquad T \text{ maps } C_0^\infty(\Omega) \text{ into } C_0^\infty(\Omega).$$

For example, (14) follows from the fact that T is the inverse of $(I+\Delta)$ and for any $\phi\in C_0^\infty(\Omega)$ the quation $(I+\Delta)u=\phi$ has a solution in $C_0^\infty(\Omega)$. For the proof of (14), one needs to extend T on differential forms, see [3].

NOTE. The following LEMMA 2 together with the comment following (12) imply that (*) gives the least eigenvalue λ satisfying

$$\Delta F=\lambda F\ \ (F\in H_0^1(\Omega)),$$

when the domain Ω is vanishing at ∞.

LEMMA 2. *Let Ω be an open set which vanishes at ∞ (i.e. Rellich's Lemma is valid) and let λ be an eigenvalue of Δ satisfying*

$$\Delta F=\lambda F\ \ (F\in H_0^1(\Omega)).$$

Then there is a sequence $\{\phi_n\}\subset C_0^\infty(\Omega)$ such that

$$\lambda=\lim_{n\to\infty}\left(\frac{\int(\nabla\phi_n,\ \nabla\phi_n)dx}{\int(\phi_n,\ \phi_n)dx}\right).$$

Proof. Since the Rellich's Lemma is valid, we can use LEMMA 1, (13) and (14) in the proof of LEMMA 2. By the definitions of T and Δ we have

$$TF=(1/1+\lambda)F.$$

Since $F\in H_0^1(\Omega)$, there is a sequence $\{\phi_n\}\subset C_0^\infty(\Omega)$ such that

$$\lim_{n\to\infty}\phi_n=F \text{ in } H_0^1(\Omega).$$

Hence we have

$$\begin{aligned}(F,\ (I+\Delta)F)&=(1+\lambda)^2(F,\ TF)\\&=(1+\lambda)^2\lim_{n\to\infty}(\phi_n,\ T\phi_n)\\&=(1+\lambda)^2\lim_{n\to\infty}(T\phi_n,\ (I+\Delta)T\phi_n).\end{aligned}$$

Since $T\phi_n\in C_0^\infty(\Omega)$, appealing to the Green's formula we have

$$\begin{aligned}(F,\ (I+\Delta)F)&=(1+\lambda)^2\lim_{n\to\infty}[(T\phi_n,\ T\phi_n)+(\nabla T\phi_n,\ \nabla T\phi_n)]\\&=(1+\lambda)^2\lim_{n\to\infty}[(TF,\ TF)+(T\nabla\phi_n,\ T\nabla\phi_n)]\\&=(1+\lambda)^2[(TF,\ TF)+(T\nabla F,\ T\nabla F)]\\&=(F,F)+(1+\lambda)^2(\nabla TF,\ \nabla TF)\\&=(F,F)+(\nabla F,\ \nabla F),\end{aligned}$$

where the second and the third eaquality follows from (13). Hence we have

$$\lambda\int(F,F)dx=\int(\nabla F,\ \nabla F)dx.$$

Since $\phi_n\to F$ in H_0^1, we know that

$$\lim_{n\to\infty}\lambda\int(\phi_n,\ \phi_n)dx=\lim_{n\to\infty}\int(\nabla\phi_n,\ \nabla\phi_n)dx.$$

This proves the Lemma.

2. Main Results

We need the following Lemma in the proof of our theorem. It is well known but for the convenience of the reader we give a proof.

LEMMA 3. *Let A, B be open sets in R^n, $F \in L^2(A)$ and $G \in L^2(B)$. Then $F(y)G(y-x)$ is contained in $L^2(A \cap B_x)$ for almost every x. Let $f_n \in C_0^\infty(A)$, $g_n \in C_0^\infty(B)$, $\lim_{n\to\infty} f_n = F$ in $L^2(A)$ and $\lim_{n\to\infty} g_n = G$ in $L^2(B)$. Then there is a subsequence of $\{f_n(y)g_n(y-x)\}$ which converges to $F(y)$ $G(y-x)$ in $L^2(A \cap B_x)$ almost every x.*

Proof. Let F, G, f_n and g_n be given in the above. Since

$$\int |F(y)G(y-x)|^2 dxdy = \int F^2(y)\,dy \cdot \int G^2(y)\,dy,$$

the Fubini's theorem implys that $F(y)G(y-x)$ is contained in $L^2(A \cap B_x)$ almost every x. Now extend F (and f_n) to R^n by $F(x)=0$ (and $f(x)=0$) for $x \notin A$, and similarly for G (and g). By the continuity of the convolution in L^1 space we have

$$\lim_{n\to\infty} \int |F^2(y)G^2(y-x) - f_n^2(y)g_n^2(y-x)|\,dxdy = 0. \tag{15}$$

One can choose a subsequence $\{f_k\}$ of $\{f_n\}$ ($\{g_k\}$ of $\{g_n\}$) such that $f_k \to F(g_k \to G)$ almost every y in L^∞. For simplicity of notations we consider sequences $\{f_n\}$ and $\{g_n\}$ converging to F and G respectively. Appealing to the Fatou's Lemma we have followings;

$$\begin{aligned} &\lim_{n\to\infty} \int |F^2(y)G^2(y-x) - f_n^2(y)g_n^2(y-x)|\,dxdy \\ &= \lim_{n\to\infty} \int [\ |F^2(y)G^2(y-x) - f_n^2(y)g_n^2(y-x)|\,dy]\,dx \\ &\geq \int [\lim_{n\to\infty}\ |F^2(y)G^2(y-x) - f_n^2(y)g_n^2(y-x)|\,dy]\,dx. \end{aligned}$$

By (15) we have

$$\lim_{n\to\infty} \int |F^2(y)G^2(y-x) - f_n^2(y)g_n^2(y-x)|\,dy = 0,$$

almost every x. This proves that $f_n(y)g_n(y-x)$ is converging to $F(y)$ $G(y-x)$ in $L^2(A \cap B_x)$ almost every x.

The following Lemma will give almost all of the results to be proved.

LEMMA 4. *Let A, B be open in R^n, $F \in H_0^1(A)$, $G \in H_0^1(B)$ and $\|F\| = \|G\| = 1$ in $L^2(R^n)$. Then we have the following relations:*

(16) $F(y)G(y-x)$ *is contained in* $H_0^1(A\cap B_x)$,

almost every x,

(17) $\nabla[F(y)G(y-x)]=\nabla F(y)\cdot G(y-x)+F(y)\cdot\nabla G(y-x)$,

almost every x,

(18)
$$\int|\nabla[F(y)G(y-x)]|^2dxdy$$
$$=\int|\nabla F(y)|^2dy+\int|\nabla G(y)|^2dy.$$

Proof. Since $F\in H_0^1(A)$ $(G\in H_0^1(B))$ one can choose $f_n\in C_0^\infty(A)$ $(g_n\in C_0^\infty(B))$ such that $f_n\to F$ (and $g_n\to G$) in H_0^1. By the Lemma 3, we can choose a subsequence $\{f_k(y)g_k(y-x)\}$ of $\{f_n(y)g_n(y-x)\}$ such that $f_k(y)g_k(y-x)$ and it's first derivatives converge to $F(y)G(y-x)$ and it's first derivatives respectively almost every x. This implies that $F(y)G(y-x)\in H_0^1(A\cap B_x)$ almost every x. Let $\phi\in C_0^\infty(R^n)$ and $\{f_k(y)$ $g_k(y-x)\}$ be the sequence given in the above then almost every x we have

$$\int F(y)G(y-x)\cdot\nabla\phi(y)dy$$
$$=\lim_{n\to\infty}\int f_k(y)g_k(y-x)\cdot\nabla\phi(y)dy$$
$$=\lim_{n\to\infty}\int[\nabla f_k(y)\cdot g_k(y-x)+f_k(y)\cdot\nabla g_k(y-x)]\phi(y)dy$$
$$=\int[\nabla F(y)\cdot G(y-x)+F(y)\cdot\nabla G(y-x)]\phi(y)dy.$$

This proves that (17) is true in the sense of distribution. For the proof of (18), consider the following equations

$$|\nabla[F(y)G(y-x)]|^2$$
$$=|F(y)\cdot\nabla G(y-x)+\nabla F(y)\cdot G(y-x)|^2$$
$$=F^2(y)|\nabla G(y-x)|^2+|\nabla F(y)|^2G^2(y-x)$$
$$+2F(y)\cdot\nabla F(y)\cdot G(y-x)\cdot\nabla G(y-x).$$

By a simple calculation as the above one can verify that $(\nabla G^2)(y-x)=2G(y-x)\cdot\nabla G(y-x)$, in the sense of distribution. Hence we have

$$\int F(y)\cdot\nabla F(y)\cdot G(y-x)\cdot\nabla G(y-x)\,dxdy=0.$$

Since F and G have the unit norm in L^2, the above two equations imply (18).

Now we prove the theorem

THEOREM. *Let A be a domain (connected open) vanishing at ∞ and B be a bounded domain. Let $\lambda(A)$ (and $\lambda(B)$) be the lowest eigenvalues of Δ on $H_0^1(A)$ (and $H_0^1(B)$). Then there is a set X of positive measure in R^n such that*

$$\lambda(A\cap B_x)<\lambda(A)+\lambda(B) \quad x\in X.$$

Proof. Let $\lambda(A)$ and $\lambda(B)$ be the eigenvalues given in the hypothesis. Then by the (Note) given in section 1, one can choose $F\in H_0^1(A)$ and $G\in H_0^1(B)$ such that

$$\Delta F=\lambda(A)F \text{ and } \Delta G=\lambda(B)G.$$

Set

$$\alpha(x)=\int |F(y)G(y-x)|^2dy, \tag{19}$$

$$\beta(x)=\int |\nabla[F(y)G(y-x)]|^2dy. \tag{20}$$

We may assume that $\|F\|=\|G\|=1$ in $L^2(R^n)$. Then by the Lemma 3 we have

$$\int \alpha(x)\,dx=1, \tag{21}$$

$$\int \beta(x)\,dx=\lambda(A)+\lambda(B). \tag{22}$$

By the same Lemma we have

$$F(y)G(y-x)\in H_0^1(A\cap B_x),$$

almost every x.

For the proof we need the inequality of Faker [3] and Krahn [4]; if β^n is the lowest eigenvalue of the n-dimmensional unit ball then

$$\lambda(A)\geq\beta^n/[\mathrm{Vol}(A)]^{2/n}.$$

Put

$$(23)\qquad Y=\{y\,|\,F(y)\neq 0,\ \ y\in A\}.$$

Then the strong maximal principle (see [2]) says that Y is not bounded in R^n. Since Y is vanishing at ∞, we have

$$(24)\qquad \mathrm{Vol}\{y\,|\,y\in A\cap B_x\}\to 0 \text{ as } x\to\infty.$$

Choose $\delta>0$ such that

$$\lambda(A)+\lambda(B)=\beta^n/(\delta)^{2/n}.$$

By (24) one can choose a t such that $|x|>t$ implies $\mathrm{Vol}(A\cap B_x)<\delta$.

Put

$$M=\{x\,|\,F(y)G(y-x)\neq 0,\ \ |x|>t\}.$$

Then by the Lemma 4 we can choose $N\subseteq M$ such that N and M have the same measure in R^n and

$$F(y)G(y-x)\in H_0^1(A\cap B_x)\ \ (x\in N).$$

It is clear that when $x\in N$ we have

$$\lambda(A\cap B_x)>\lambda(A)+\lambda(B)$$

and

$$(25)\qquad \beta(x)>(\lambda(A)+\lambda(B))\alpha(x).$$

By (25) we have

$$\int_N[(\lambda(A)+\lambda(B))\alpha(x)-\beta(x)]dx<0.$$

Since

$$\int_{R^n}[(\lambda(A)+\lambda(B))\alpha(x)-\beta(x)]dx=0,$$

we can find a subset $X\subset R^n$ with positive measure such that

$$\beta(x)>(\lambda(A)+\lambda(B))\alpha(x)\ \ (x\in X).$$

This proves the Theorem.

Following Lieb [5], we give an example that shows the Theorem 2 is false for the general unbounded two domains.

EXAMPLE. Let $A=\{(x,y)\mid -\infty<y<\infty,\ 0<x<\pi\}$ and $B=\{(x,y)\mid -\infty<x<\infty,\ 0<y<\pi\}$. Then we have

$$2=\lambda(A)+\lambda(B)=\lambda(A\cap B_x)$$

for each $x\in R^2$, and $\lambda(A\cap A_x)$ takes any real from 2 to ∞.

The above example shows that the value of $\lambda(A\cap B_x)$ depends on the locations of A and B in R^n.

References

1. Faker, C.: *Bewis das unter allen homogenen Membranes von gleicher Flächer und gleicher spanung die kreisformige den tiefsten Grundton gibt*, Sitzunsker. Byer. Akad. der Wiss. Math. Phys., Munich 1923, 169–172.
2. Gilbarg, D. Trudinger, N.S.: *Elliptic partial differential equations of second order*, Springer-Verlag, Belin, 1977.
3. Griffiths, P. and Harris, J.: *Principles of algebraic geometry*, John Willy and Sons, New York, 1978.
4. Krahn, F.: *Über ein von Reyleigh formulierte Minimaleigenshaft des Kreises*, Math. Ann. 94, 97–100 (1925).
5. Lieb, H.L.: *On the lowest eigenvalue of the Laplacian for the intersection of two domains*, Invent. Math. 74, 441–448 (1983).
6. Schechter, M.: *Modern methods in partial differential equations*, McGraw Hill, New York, 1977.

Seoul National University
Seoul, Korea