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STRUCTURES OF AF ALGEBRAS

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1. Introduction

1. 1. An *approximately finite dimensional C^*-algebra*, or in short an AF algebra, is a C^*-algebra which is the norm closure of an increasing sequence of finite dimensional C^*-algebras. AF algebras were introduced by Bratteli [1]. In [1], Bratteli developed a device, the so-called "Bratteli diagram", to investigate various properties of AF algebras. Earlier more restrictive classes of AF algebras, namely the norm closure of an increasing sequence of full matrix algebras, had been studied by Glimm [8] for unital case (UHF algebra) and by Dixmier [5] for non-unital case (matroid C^*-algebra).

1. 2. The main distinction between AF algebras and their immediate predecessors (UHF algebras and matroid C^*-algebras) is that the latter algebras are always simple, while this is not the case for the former in general. Since AF algebras are relatively simple to handle without being trivial, they have served as testing grounds for conjectures and provided many fine examples for the theory of C^*-algebras. By use of variants of Bratteli diagrams, Lazar and Taylor [10] characterized AF algebras that are liminal, postliminal, antiliminal and with continuous trace.

1. 3. Elliott [7] found another method, namely partially ordered group, to study AF algebras. This method turned out to be very powerful and indeed Elliott showed that the (stable) isomorphism classes of AF algebras are completely determined by its dimension group (see also Effros [6]).

1. 4. Our purpose of this note is to study some properties of AF algebras in terms of their dimension groups. More precisely, we characterized liminal AF algebras in terms of their K_0-groups and we obtain more complete pictures of liminal AF algebras [2]. Using these results

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we also obtain a number of results about homogeneous AF algebras. Our approach and method in this note are different from those of Lazar and Taylor [10] and we improve a number of their results.

2. AF algebras and K-theory

2. 1. We briefly discuss the construction of K_0-group of a general C^*-algebra A. More details can be found in several sources (see for example [3], [6], [11]). Let $M_n(A)$ denote the set of $n\times n$ matrices of entries from A. Then it is easy to see that $M_n(A)$ is actually a C^*-algebra with the usual matrix addition, matrix multiplication and with the natural C^*-norm. Let $\mathrm{Proj}_n(A)$ be the set of all projections in $M_n(A)$. Then we have natural injections

$$\mathrm{Proj}_n(A)\to\mathrm{Proj}_{n+1}(A)$$

defined by $e\to e\oplus 0$. We define an equivalence relation on $\mathrm{Proj}_n(A)$ by letting $e\simeq f$ if there is a partial isometry u in $M_n(A)$ such that $u^*u=e$ and $uu^*=f$. Then it is easy to see that the inclusions respect this equivalence relation on $M_n(A)$. We denote $D_n(A)$ the set of all equivalence classes. Then we have a system of sets

$$D_n(A)\to D_{n+1}(A).$$

For any e, f in $D_n(A)$ we define

$$e+f=(e\oplus 0)+(0\oplus f)$$

in $D_{2n}(A)$. Then it is routine to check that the inductive limit $D(A)$ of $D_n(A)$ with the addition defined as above gives us a semigroup $D(A)$. Finally the Grothendieck group for $D(A)$ is called the K_0-group $K_0(A)$. It is easy to see that $K_0(M_n(\boldsymbol{C}))\simeq\boldsymbol{Z}$, the integer group, and that the inclusion $i: M_n(\boldsymbol{C})\to M_{np}(\boldsymbol{C})$ induces $i_*: K_0(M_n(\boldsymbol{C}))\to K_0(M_{np}(\boldsymbol{C}))$ and $i_*(1)=p$. It is not hard to see that K_0 is a covariant functor from the category of C^*-algebras and $*$-homomorphisms to the category of abelian groups and homomorphisms. Furthermore, for the C^*-direct sum of two C^*-algebras A and B, we have $K_0(A\oplus B)\simeq K_0(A)\oplus K_0(B)$.

Thus for a finite demensional C^*-algebra A, say $A\simeq M_{n_1}(\boldsymbol{C})\oplus\cdots\oplus M_{n_k}(\boldsymbol{C})$, we have $K_0(A)\simeq\boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}$($k$ copies of $\boldsymbol{Z}$).

We next state another important property of K_0-groups of C^*-algebras.

2. 2. THEOREM. *Suppose that A is a C^*-algebra and that $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of C^*-subalgebras with $\overline{UA_n}=A$. Then we have*

$$K_0(A) \cong \varinjlim K_0(A_n)$$

Proof. See [6, Proposition 8. 1.] for example.

2. 3. A partially ordered group is called a *dimension group* if it is a direct limit of free abelian group $\boldsymbol{Z}^n$ with the coordinate ordering (see [6], [9] for more details). Thus the K_0-group of AF algebra is a dimension group. A subgroup H of a dimension group G is an *order ideal* if H is directed and $0<x<y$, $y \in H$ implies $x \in H$. For AF algebra A and a closed two sided ideal I of A, $K_0(I)$ is an order ideal of $K_0(A)$. And furthermore any order ideal of $K_0(A)$ is the dimension group of some closed two sided ideal of A (see [9] for example).

3. Compact ideal of AF algebra and liminal AF algebra.

3. 1. Let $L(H)$ be the algebra of all bounded operators on a separable Hilbert space H. Throughout this paper $K(H)$ denotes the set of all compact operators on H. If H is infinite-dimensional, then $K(H)$ is the only norm closed proper two sided ideal of $L(H)$. As an ideal, $K(H)$ is generated by all finite projections in $L(H)$.

In this section we study the ideal generated by finite projection of AF algebra A. Our definition of finite dimensionality of projection of A resembles that of projection in $L(H)$. A projection p in an AF algebra A is called *minimal in* A if there is no proper nonzero subprojection of p in A. A projection p is called *finite in* A if it can be written as a finite sum of mutually orthogonal minimal projections in A. If a projection is not finite in A, then it is called *infinite in* A.

3. 2. Let A_n be an increasing sequence of finite dimensional C^*-subalgebras of A with $\overline{UA_n}=A$. Let p be a projection in A. Then there exists a projection q in UA_n which is equivalent to p. Since the properties under investigation (K_0-group, ideal, finiteness, etc.) are invariant under the equivalence relation of projections, we may assume throughout the rest of this paper that any projection p in A actually belongs to some A_n. Let $[p]_n$ denote the element of $K_0(A_n)$ determined by p if p belongs to A_n. The $(0, \cdots, 0, [p]_n, [p]_{n+1}, \cdots)$ represent an element of $K_0(A)$. We say that a projection p in A_n determines an *eventually constant sequence*

if there exists a natural number k such that for all $m\geqq k$, $[p]_m$ remains constant ignoring the number of zeros in the free abelian group $K_0(A_m)$.

3. 3. DEFINITION. Let A be an AF algebra. Then the norm closed two sided ideal generated by all finite projections in A is called the *compact ideal of* A. We denote it by $K(A)$.

If H is a finite dimensional Hilbert space, then the compact ideal $K(H)$ of $L(H)$ becomes the full matrix algebra. Let $\sum_{\Lambda}\oplus K(H_\alpha)$ denote the C^*-direct sum of compact ideals $K(H_\alpha)$ over a countable index set Λ. Let $M_0=\{(x_\alpha)\mid(x_\alpha)\in\sum_{\Lambda}K(H_\alpha),\ \lim_{\alpha\to\infty}\|x_\alpha\|=0\}$. Of course, for a finite index set Λ, $M_0=\Sigma\oplus K(H_\alpha)$. Then it is easy to see that this algebra M_0 can be the compact ideal of some AF algebra. If Λ is the empty set, then of course M_0 means the trivial ideal 0. Furthermore any compact ideal of AF algebra is the above form and the following theorem is taken from [2].

3. 4. THEOREM [2]. *Let A be an AF algebra. Then the compact ideal $K(A)$ of A is *-isomorphic to some M_0.*

3. 5. Let A be an AF algebra with $\overline{UA_n}=A$, where A_n are increasing sequence of finite dimensional C^*-subalgebras of A. Let p_n be a sequence of projections such that

(i) each projection p_n is a minimal projection as a projection of A_n.

(ii) each p_{n+1} is a subprojection of p_n.

Observe that if p_n is a finite projection in A, then eventually all p_n must be the same. But if all p_n are an infinite projection in A, then we can choose p_n so that p_{n+1} is a proper subprojection of p_n. Let d_n be the number of minimal orthogonal projections in A_n each of which is equivalent to p_n. First we have the following proposition.

3. 6. PROPOSITION [2]. *Let p_n, d_n be as above. If the sequence d_n is eventually constant (i.e., there exists a natural number m such that for all $n\geqq m$ we have $d_n=d_m$), then there exists a closed two sided ideal I of A such that $\pi(p_m)=\pi(p_n)$ for all $n>m$, where π is the natural homomorphism of A onto A/I.*

3. 7. Recall that a C^*-algebra A is called *liminal* if for every irreducible representation $\pi, \pi(x)$ is compact operator for every x in A. Hence for every unital C^*-algebra A and for every irreducible representation π, $\pi(A)$

is *-isomorphic to a full matrix algebra. Thus the kernel of any irreducible representation of a liminal C^*-algebra A is a maximal closed two sided ideal of A. See Dixmier [4] for general properties of liminal C^*-algebras. The stated conditions of a sequence of projections p_n of Proposition 3. 6. are satisfied in a unital liminal AF algebras.

In what follows let p_n and d_n be the some as described in 3. 5.

PROPOSITION [2]. *Let A be a unital liminal AF algebra. Then for any sequence of projections p_n, the sequence of associated numbers d_n are eventually constant.*

3. 8. COROLLARY. *Let A be a unital liminal AF algebra. Then the compact ideal $K(A)$ of A is *-isomorphic to M_0 of which each component is a full matrix algebra.*

4. Homogeneous AF algebras

4. 1. Let α be a cardinal number. Recall the a C^*-algebra is called *homogeneous* of *degree* α if every irreducible representation of A is of dimension α. In this section we characterize homogenous AF algebras and then using our characterization we study various properties of homogeneity. First we prove the following.

PROPOSITION. *Let A be a homogeneous (not necessarily unital)* AF *algebra of degree α $(\alpha<\infty)$. Let p_m be a sequence of projections as in* 3. 5 *and d_m the sequence of associate numbers. Then $\lim_{m\to\infty} d_m=\alpha$.*

Proof. First we consider the untial case. By Proposition 3. 7, the sequence d_m is eventually constant. Suppose that there exists an associated sequence d_m such that $\lim_{m\to\infty} d_m=l\neq\alpha$. If the corresponding projections p_m are finite projections in A, then by Corollary 3. 8 A is *-isomorphic to the C^*-direct sum of $M_l(C)$ and some B, which is a contradiction to the fact that A is homogeneous of degree α. Thus p_m must be all infinite projections. Then by the same arguement in the proof of Proposition 4. 1 of [2] we get a representation of degree l, which is again contradiction. Thus $\lim_{m\to\infty} d_m=\alpha$.

Next we consider the non-unital case. Suppose that p_m are finite projections in A. Then there are two possibilities. The first possibility is that

$\lim_{m\to\infty} d_m$ exists. In this case the limit must be the same number α, otherwise we obtain the same contradiction as in the unital case. The second possibility is that the limit does not exist. In this case $A \cong K(H) \oplus B$ (C^*-direct sum). Again we get a contradiction to the homogeneity of A. Now suppose that all p_n are infinite in A and that $\lim_{n\to\infty} d_n \neq \alpha$. Suppose that the limit exists but does not agree to α. This case reduces to the unital case. If the limit does not exist, then either A is not liminal or A has a representation of $\aleph_0$, which is a contradiction. This compleses the proof.

4. 2. PROPOSITION. *Let A be an AF algebra. If there exists a natural number α such that for any sequence of projections p_n as in* 3. 5 $\lim_{n\to\infty} d_n = \alpha$, *then A is homogeneous of degree α.*

Proof. Let π be any irreducible representation of A and I be its kernel. It is plain to see that A/I does satisfy the same condition of the proposition. Thus unless A/I is *-isomorphic to a full matrix algebra, A/I would not be irreducible. Therefore A/I is *-isomorphic to $M_\alpha(C)$, which completes the proof.

4. 3. COROLLARY. *Let A be an AF algebra. Then A is homogeneous of degree α ($\alpha < \infty$) if and only if for any decreasing sequence of projections p_n as in* 3. 5, $\lim_{n\to\infty} d_n = \alpha$.

4. 4. COROLLARY. *Let A be a homogeneous AF algebra of degree α. Then any ideal of A and any quotient C^*-algebra of A are both homogeneous of degree α.*

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