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# A TUBE FORMULA IN PRODUCT RIEMANNIAN MANIFOLDS

SUNG YUN LEE

In 1973 Gray [1] proved a product formula for a geodesic ball in product Riemannian manifolds. We generalize his result in this article. In fact we derive a product formula for a tube in product Riemannian manifolds (Theorem 4).

Let $M$ be a smooth Riemannian manifold of dimension $n$ with metric tensor $\langle , \rangle$. Denote by $\mathfrak{X}(M)$ the smooth vector fields on $M$, and let $\nabla$ and $R$ be the Riemannian connection and curvature tensor of $M$. Here $\nabla$ and $R$ are given by

$$2\langle \nabla_X Y,\ Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X,\ [Y, Z]\rangle - \langle Y,\ [X, Z]\rangle + \langle Z,\ [X, Y]\rangle,$$

$$R_{XY}Z = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]Z, \text{ for } X, Y, Z \in \mathfrak{X}(M).$$

Let $P$ be a $p$-dimensional embedded submanifold with compact closure in $M$. Briefly we put $P \subset M$. We assume that $P$ and $M$ are analytic Riemannian manifolds. We now give a definition of fermi coordinates which describes the geometry of $M$ in a neighborhood of $P$. We denote by $\nu$ the normal bundle of $P$ in $M$. The exponential map $\exp_\nu$ of the normal bundle maps a neighborhood of the zero section of $\nu$ into $M$. Let $p \in P$ and let $E_{p+1}, \cdots, E_n$ be orthonormal sections of $\nu$ defined near $p$. Let $(y_1, \cdots, y_p)$ be a coordinate system in a neighborhood $V$ of $p$ in $P$.

DEFINITION. The *Fermi coordinates* $(x_1, \cdots, x_n)$ of $P$ at $p$ (relative to $(y_1, \cdots, y_p)$ and $E_{p+1}, \cdots, E_n$) are given by

$$x_a(\exp_\nu(\sum_{j=p+1}^{n} t_j E_j(q))) = y_a(q), \quad a = 1, \cdots, p,$$

$$x_i(\exp_\nu(\sum_{j=p+1}^{n} t_j E_j(q))) = t_i, \quad i = p+1, \cdots, n, \quad \text{for } q \in V.$$

Let $\omega$ be the volume element on $M$ and let $(x_1, \cdots, x_n)$ be a system of

Fermi coordinates of $P\subset M$ at $p$ such that $\omega\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)>0$. We put $\frac{\partial}{\partial x_k}=X_k$, $1\leqq k\leqq n$, and

$$\omega(X_1, \cdots, X_n)=\omega_{1\cdots n}.$$

The *tube* $T(P,r)$ *of radius* $r$ about $P$ in $M$ is the set

$$T(P,r)=\{\exp_p(u)\,|\,p\in P,\ u\in P_p^{\perp},\ \|u\|\leqq r\}.$$

We also put the tubular hypersurface

$$P_r=\{m\in T(P,r)\,|\,d(m,P)=r\}.$$

Here we assume that $r$ is less than or equal to the distance of $P$ to its nearest focal point so that for $m\in T(P,r)-P_r$ there exists a unique geodesic from $m$ to $P$ meeting $P$ orthogonally. We set

$$V_P{}^M(r)=n\text{-dimensional volume of } T(P,r),$$
$$A_P{}^M(r)=(n-1)\text{-dimensional volume of } P_r.$$

The following basic lemmas are well-known.

LEMMA 1. *We have*

$$\int_0^r A_P{}^M(r)\,dr=V_P{}^M(r).$$

LEMMA 2. *We have*

$$A_P{}^M(r)=\int_P\int_{S^{n-p-1}(r)}\omega_{1\cdots n}(\exp_p u)\,du\,dP,$$

*where* $S^{n-p-1}(r)$ *is the* $(n-p-1)$*-dimensional sphere of radius* $r$ *in* $P_p^{\perp}$, $du$ *is the corresponding volume element, and* $dP$ *is the volume element of* $P$.

For proofs see [2].

Let $P\subset M$, $Q\subset N$ and put $\dim P=p$, $\dim Q=q$, $\dim M=m$, $\dim N=n$. Then the *Riemannian product* $P\times Q$ is a $(p+q)$-dimensional submanifold of the Riemannian product manifold $M\times N$. Let $\omega_1$ and $\omega_2$ be the volume elements of $M$ and $N$ respectively.

Then $\omega_1\wedge\omega_2$ is the volume element of $M\times N$. Let $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ be a Fermi coordinates of $P\times Q\subset M\times N$ at $(p,q)\in P\times Q$ such that $(x_1, \cdots, x_m)$ is a system of Fermi coordinates of $P\subset M$ at $p\in P$ and $(y_1, \cdots, y_n)$ is a system of Fermi coordinates of $Q\subset N$ at $q\in Q$. We put

$\frac{\partial}{\partial x_A}=X_A,\ 1\leqq A\leqq m,\ \frac{\partial}{\partial y_B}=Y_B,\ 1\leqq B\leqq n,$ and write

$$\omega_1(X_1,\cdots,X_m)=(\omega_1)_{1\cdots m};\ \omega_2(Y_1,\cdots,Y_n)=(\omega_2)_{1\cdots n};$$
$$(\omega_1\wedge\omega_2)(X_1,\cdots,X_m,\ Y_1,\cdots,Y_n)\ =(\omega_1\wedge\omega_2)_{1\cdots m,1\cdots n}.$$

Then we have

LEMMA 3.

$$(\omega_1\wedge\omega_2)_{1\cdots m,1\cdots n}=(\omega_1)_{1\cdots m}(\omega_2)_{1\cdots n}.$$

Gray and Vanhecke [3] gave an effective method determining the power series expansion in Fermi coordinates of an analytic covariant tensor field. We now define $A_P{}^M(r)$ by Lemma 2

$$A_P{}^M(r)=\int_P\int_{S^{n-p-1}(r)}\omega_{1\cdots n}(\exp_\nu u)\,du\,dP$$

for all values of $r$ provided we use the power series expansion in Fermi coordinates for $\omega_{1\cdots n}$. If $r$ is less than or equal to its nearest focal point then $A_P{}^M(r)$ is the $(n-1)$-dimensional volume of $P_r$. We also define

$$\tilde{A}_P{}^M(s)=\int_0^\infty e^{-s^2t^2}A_P{}^M(t)\,dt.$$

The power series for $e^{-s^2t^2}A_P{}^M(t)$ can be integrated term by term. Thus $\tilde{A}_P{}^M(s)$ can always be formally defined.

THEOREM 4. *Let $P\subset M$, $Q\subset N$ so that $P\times Q\subset M\times N$. Then we have*

$$\tilde{A}_{P\times Q}{}^{M\times N}(s)=\tilde{A}_P{}^M(s)\tilde{A}_Q{}^N(s).$$

*Proof.* We first recall the following.

LEMMA 5. *Let $(x_1,\cdots,x_n)$ be a system of Fermi coordinates for $P\subset M$. We put*

$$r^2=\sum_{i=p+1}^n x_i^2.$$

*Then $r(m)=d(m,P)$.*

For a proof see [2].

Next we choose a system of Fermi coordinates $(x_1,\cdots,x_m,\ y_1,\cdots,y_n)$

of $P\times Q\subset M\times N$ at $(p, q)\in P\times Q$. We write $r_1{}^2=\sum_{i=p+1}^{m} x_i^2$, $r_2{}^2=\sum_{j=q+1}^{n} y_j^2$, $r^2=r_1{}^2+r_2{}^2$. Let $\nu$ be the normal bundle of $P\times Q$ in $M\times N$ and $\nu_1$ (resp. $\nu_2$) be the normal bundle of $P$ (resp. $Q$) in $M$ (resp. $N$).

Then we have from Lemma 3

$$(\omega_1\wedge\omega_2)_{1\cdots m,1\cdots n}(\exp_\nu u)=(\omega_1)_{1\cdots m}(\exp_{\nu_1}u_1)\,(\omega_2)_{1\cdots n}(\exp_{\nu_2}u_2),$$

where $u\in(P\times Q)^\perp$ can be written $u=(u_1, u_2)$ with $u_1\in P_p{}^\perp$, $u_2\in Q_q{}^\perp$.

Therefore we have

$$\begin{aligned}
\tilde{A}_{P\times Q}{}^{M\times N}(s) &= \int_0^\infty e^{-s^2r^2}A_{P\times Q}{}^{M\times N}(r)\,dr\\
&=\int_0^\infty\int_{P\times Q}\int_{S^{m+n-p-q-1}(r)} e^{-s^2r^2}(\omega_1\wedge\omega_2)_{1\cdots m,1\cdots n}(\exp_\nu u)\,du\,d(P\times Q)\,dr\\
&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_P e^{-s^2r_1{}^2}(\omega_1)_{1\cdots m}(\exp_{\nu_1}u_1)\,dP\,dx_{p+1}\cdots dx_m\\
&\quad\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_Q e^{-s^2r_2{}^2}(\omega_2)_{1\cdots n}(\exp_{\nu_2}u_2)\,dQ\,dy_{q+1}\cdots dy_n\\
&=\int_0^\infty e^{-s^2r_1{}^2}A_P{}^M(r_1)\,dr_1\int_0^\infty e^{-s^2r_2{}^2}A_Q{}^N(r_2)\,dr_2\\
&=\tilde{A}_P{}^M(s)\tilde{A}_Q{}^N(s).
\end{aligned}$$

REMARK. If $P$ and $Q$ are points then the above theorem reduces to a product formula for a geodesic ball.

## References


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Korea Institute of Technology
Taejeon, Chung-Nam, 300-31
Korea