Comm. Korean Math. Soc. 1 (1986)

ON TOPOLOGICAL STRUCTURE OF A CERTAIN SUBMANIFOLD IN R^{n+2}

YONG BAI BAIK AND YANG JAE SHIN

1. Introduction

This is a kind of reports which is devoted to classifying certain submanifold of codimension 2 immersed in a Euclidean space R^{n+2}. As is well known, if for a compact connected hypersurface M in a Euclidean space R^{n+1} the Gauss curvature never vanishes on M, then the second fundamental form of M is definite everywhere on M and moreover M is homeomorphic to a sphere.

On the other hand, since a Riemannian submanifold of positive curvature has the Gauss curvature where never vanishes, it seems to be interesting to investigate that for an n-dimensional compact Riemannian manifold of positive curvature immersed isometrically in R^{n+2} whether the property stated above in the hypersurface is valid or not. This problem was treated by Bishop [2], Gallot-Meyer [6], Meyer [12] and Weinstein [15]. It has been almost completely classified by Moore [14]. Moore proved that if M is of positive curvature, then M is a homotopy sphere. This result is generalized by Baldin and Mercuri [1] in the case of non-negative curvature, which is stated as follows: If M is of non-negative curvature, then M is either a homotopy sphere or diffeomorphic to a product of two spheres.

The purpose of this paper is to verify the particular case of the result due to Baldin and Mercuri from a different point of view. In the last section we prove the following:

THEOREM. *Let M be an $n(\geqq 3)$-dimensional compact connected and oriented Riemannian manifold of non-negative curvature. If there is a point x on M at which all sectional curvatures are positive and if M is*

*This research was supported by Korea Science and Engineering Foundation 84-85.

isometrically immersed in a Euclidean space R^{n+2}, *then* M *is an integral homological sphere.*

2. Preliminaries

Let V and W be real vector spaces of finite dimensions n and p respectively, and σ be a symmetric bilinear map of $V\times V$ into W. Suppose $n\geqq 2$ and W has an inner product $\langle\ ,\ \rangle$. A vector u in $V_0=V-\{0\}$ is said to be *asymptotic* if $\sigma(u,u)=0$ holds. Define *the associated curvature form* $R_\sigma: \Lambda^2V\times\Lambda^2V$ to R by

(2.1) $R_\sigma(u\wedge v,\ w\wedge z)=\langle\sigma(u,w),\ \sigma(v,z)\rangle-\langle\sigma(u,z),\ \sigma(v,w)\rangle$

for any vectors u, v, w and z in V. The map R_σ is again symmetric and hence the eigenvalues of R_σ are real. R_σ is said to be positive definite or positive semi-definite according as all eigenvalues of R_σ are positive or non-negative, respectively. A real valued map K_σ is next defined by

(2.2) $$K_\sigma(u,v)=R_\sigma(u\wedge v,\ u\wedge v)$$

whenever $u\wedge v\neq 0$. The map K_σ is said *to be positive definite* or *positive semi-definite* according as $K_\sigma(u,v)$ is positive or non-negative for linearly independent vectors u and v in V_0, respectively. Consider the following conditions for the linear map:

(a) R_σ is positive semi-definite.

(b) K_σ is positive semi-definite.

(c) There exists an orthonormal basis $\{\xi_{n+1},\cdots,\xi_{n+p}\}$ for W in such a way that the real valued function h_α on $V\times V$ defined by $h_\alpha(u,v)=\langle\sigma(u,v),\ \xi_\alpha\rangle$ is non-negative for any index $\alpha=n+1,\ \cdots,\ n+p$.

LEMMA 2.1. (1) (a)$\to$(b). (2) (c)$\to$(a). (3) *In particular, if* $p=2$, *the conditions above are all equivalent.*

Proof. The assertion (1) is trivial. Suppose that the condition (c) holds. By making use of the function h_α for an orthonormal basis $\{\xi_\alpha\}$, an image of σ is given by $\sigma(u,v)=\sum_\alpha h_\alpha(u,v)\xi_\alpha$. Then we have

$$R_\sigma(u\wedge v,\ w\wedge z)=\sum\{h_\alpha(u,w)h_\alpha(v,z)-h_\alpha(u,z)h_\alpha(v,w)\}.$$

For a fixed index α, real valued function R_α is given by

(2.3) $R_\alpha(u\wedge v,\ w\wedge z)=h_\alpha(u,w)h_\alpha(v,z)-h_\alpha(u,z)h_\alpha(v,w)$, and we have

$$R_\sigma = \sum_\alpha R_\alpha \tag{2.4}$$

In order to prove that R_σ is positive semi-definite, it suffices to show that all the map R_α are positive semi-definite. For fixed index α, let $\{u_1, \cdots, u_n\}$ be an orthonormal basis for V which diagonalizes the function h_α; namely, $h_\alpha(u_i, u_j) = \lambda_i \delta_{ij}$. Here and in the sequel, indices i and j run over the range $\{1, 2, \cdots, n\}$ and an index α run over the range $\{n+1, \cdots, n+p\}$, unless otherwise stated. Then $\lambda_i \geq 0$ for all indices i, because h_α is positive semi-definite. Since the inner product $\langle\, , \rangle$ of $\Lambda^2 V$ is by definition

$$\langle u \wedge v,\ w \wedge z \rangle = \langle u, w \rangle \langle v, z \rangle - \langle u, z \rangle \langle v, w \rangle,$$

then the definition (2.3) of the function R_α implies

$$R_\alpha(u_i \wedge u_j,\ u_k \wedge u_l) = \lambda_i \lambda_j \langle u_i \wedge u_j,\ u_k \wedge u_l \rangle.$$

It means that $\{u_i \wedge u_j : i < j\}$ forms an orthonormal basis for $\Lambda^2 V$ which diagonalizes R_α with eigenvalues $\lambda_i \lambda_j$ ($\geqq 0$). So R_α is positive semi-definite.

In the case where $p=2$, it only remains to prove that the condition (b) implies the condition (c). Suppose that the map K_σ is positive semi-definite. Then for all pairs (u, v) of linearly independent vectors. we have

$$K_\sigma(u, v) = \langle \sigma(u, u),\ \sigma(v, v) \rangle - \|\sigma(u, v)\|^2 \geqq 0, \tag{2.5}$$

where $\|\ \|$ means the norm for the vector space W. Now there might exist a non-asymptotic vector u_0 in V_0. indeed, suppose that any vector u in V_0 is asymptotic. Then $h_\alpha(u, u)$ must be equal to zero, because of $h_\alpha(u, u) = \langle \sigma(u, u),\ \xi_\alpha \rangle$ for any orthonormal basis $\{\xi_\alpha\}$ for W. If this case can be regarded as the special one of positive semi-definiteness, then it is nothing but the condition (c). Choose an orientation for W, and for fixed vector u_0 and any vector u in V_0 let $\theta(u)$ denote an angle from $\sigma(u_0, u_0)$ to $\sigma(u, u)$. $\theta(u)$ is defined only module 2π but it follows from (2.5) that θ is a continuous function of V_0 into the closed interval $[-\pi/2, \pi/2]$. For a unit sphere S of V centered with origin, the restriction of θ to S is also continuous, so it must attain its maximum

θ_1 and minimum θ_2. Again, taking the inequality (2. 5) into account, we get $\theta_1-\theta_2\leqq\pi/2$. Let $\bar{\theta}=(\theta_1+\theta_2)/2$, $\bar{\theta}_1=\bar{\theta}+\pi/4$ and $\bar{\theta}_2=\bar{\theta}-\pi/4$, and $\xi(\theta)$ be a unit vector in W to which the direct angle from $\sigma(u_0, u_0)$ is equal to θ. Then, by putting $\xi_{n+1}=\xi(\bar{\theta}_1)$ and $\xi_{n+2}=\xi(\bar{\theta}_2)$, $\{\xi_{n+1}, \xi_{n+2}\}$ is an orthonormal basis for W, and by choosing the angle θ_1 and θ_2 it turns out that

$$\bar{\theta}_2\leqq\theta_2\leqq\theta(u)\leqq\theta_1\leqq\bar{\theta}_1$$

for any vector u in S. This implies that the angle between $\xi(\theta(u))$ and $\xi_\alpha(\alpha=n+1, n+2)$ is less than or equal to $\pi/2$ for any u in S, and so is the angle between $\sigma(u, u)$ and ξ_α, because of $\sigma(u, u)=\|\sigma(u, u)\|\xi(\theta(u))$. Thus the forms h_α are both positive semi-definite. This concludes the proof.

For any ξ in W, a symmetric transformation A_ξ is defined by $\langle A_\xi u, v\rangle=\langle\sigma(u, v), \xi\rangle$. Assume that $p=2$, and let $\{\xi_{n+1}, \xi_{n+2}\}$ be an orthonormal basis for W. Put $A_\alpha=A_{\xi_\alpha}$ and then $A_\theta=A_{\xi(\theta)}$ for any unit vector $\xi(\theta)=\cos\theta\cdot\xi_{n+1}+\sin\theta\cdot\xi_{n+2}$. Then it turns out that

$$A_\theta=\cos\theta\cdot A_{n+1}+\sin\theta\cdot A_{n+2}. \tag{2. 6}$$

For the comparison property of the absolute value of $\det A$ the following lemma is proved. This is essentially due to Chen [3].

LEMMA 2. 2. *Let W be a 2-dimensional vector space. If the associated curvature form R_σ is positive semi-definite, then*

$$|\det A_{\pi-\theta}|\leqq|\det A_\theta|, \quad \textit{for all } \theta\in[0, \pi/2].$$

In particular, if R_σ is positive definite, then

$$|\det A_{\pi-\theta}|<|\det A_\theta|, \quad \textit{for all } \theta\in(0, \pi/2).$$

Proof. The first assertion will be only verified. By Lemma 2. 1 it turns out that there exists an orthonormal basis $\{\xi_{n+1}, \xi_{n+2}\}$ in such a way that the function is positive semi-definite, namely $\langle\sigma(u, u), \xi_\alpha\rangle\geqq0$ for any vector u in V and any index α. This means that the symmetric transformation A_α is positive semi-definite. For this basis $\{\xi_{n+1}, \xi_{n+2}\}$ the construction above of A_ξ shows that $A_0=A_{n+1}$ and $A_{\pi/2}=A_{n+2}$.

Suppose that first A_0 and $A_{\pi/2}$ are both trivial transformations, that is, they are both zero matrices in the matrix expression. Then the relation

(2.6) implies $A_\theta=0$ for all $\theta\in[0, 2\pi]$, namely, $\sigma(u, v)=0$ for any vectors u and v in V, so σ vanishes everywhere on V. So $\det A_{\pi-\theta}=\det A_\theta=0$ for any θ and the assertion is satisfied.

Next, suppose that either A_0 or else $A_{\pi/2}$ is a non-zero matrix. Since the real valued function : $\theta\to\det A_\theta$ on $[0, \pi/2]$ is a non-trivial real analytic function of θ, its zeroes must be equal to the entire definition domain $[0, \pi/2]$ or it is discrete. Consider first the case where its zeroes are equal to $[0, \pi/2]$. Since there exists a vector u' in V_0 such that $A_\theta u'=0$ for any $\theta\in(0, \pi/2)$, we get

$$\langle\sigma(u', u'),\ \cos\theta\cdot\xi_{n+1}+\sin\theta\cdot\xi_{n+2}\rangle=0,$$

so that $\langle\sigma(u', u'),\ \xi_\alpha\rangle=0$ for any α, because of $\langle\sigma(u', u'),\ \xi_\alpha\rangle\geqq0$. Hence $\langle\sigma(u', u'),\ \xi\rangle=0$ for any vector ξ in W, which implies that $A_\xi u'=0$ for any vector u' in V_0. This means that $\det A_\xi=0$. Thus under the condition that $\det A_\theta=0$ for any θ in $[0, \pi/2]$, we get $\det A_\xi=0$ for any unit vector ξ in W. This concludes the assertion.

Suppose next the set $\{\theta : \det A_\theta=0\}$ is discrete. A number θ_0 can be picked in $(0, \pi/2)$ such that $\det A_{\theta_0}\neq0$. Because A_{θ_0} can be regarded as a positive semi-definite symmetric $n\times n$ matrix, it is positive definite and diagonalizable, and there exists an orthonormal basis $\{u_1, \cdots, u_n\}$ for V such that the matrix expression of A_{θ_0} is

$$\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}=I(\lambda_1, \cdots, \lambda_n)$$

with all $\lambda_i>0$. Again any unit vector $\xi(\theta)$ for $\theta\in[0, \pi/2]$ can be expressed as $\xi(\theta)=c\xi(\theta_0)+s\xi(\pi/2)$, where $c=\cos\theta/\cos\theta_0$ and $s=\sin(\theta-\theta_0)/\cos\theta_0$. Hence $A_\theta=cA_{\theta_0}+sA_{\pi/2}=cI(\lambda_1, \cdots, \lambda_n)+sA_{\pi/2}$. For a unit matrix $I=I(1, \cdots, 1)$ and a matrix $I'=I(1/\sqrt{\lambda_1}, \cdots, 1/\sqrt{\lambda_n})$,

$$I'A_\theta I'=cI+sI'A_{\pi/2}I'.$$

Since $A_{\pi/2}$ is a symmetric positive semi-definite matrix, so is the matrix $I'A_{\pi/2}I'$. Accordingly there exists a regular $n\times n$ matrix P so that it diagonalizes the last matrix, and then

$$P^{-1}(I'A_\theta I')P=cI+sI(\mu_1, \cdots, \mu_k, 0, \cdots, 0),$$

where all $\mu_i>0$. By taking the determinant of both sides of this equation, one finds

$$\det A_\theta/\prod_{i=1}^{n}\lambda_i=\prod_{j=1}^{k}(c+s\mu_j)c^{n-k}$$

$$=(\cos\theta)^{n-k}\prod_{j=1}^{k}\{\cos\theta+\mu_j\sin(\theta-\theta_0)\}/(\cos\theta_0)^n$$

The estimate of this equation being applied, the comparison of $|\det A_\theta|$ with $|\det A_{\pi-\theta}|$ can be complete. Since λ_i, θ_0 and μ_j are all constant for this equation with respect to the variable θ, it only remains to compare $(\cos\theta)^{n-k}\prod_{j=1}^{k}\{\cos\theta+\mu_j\sin(\theta-\theta_0)$ with $(\cos(\pi-\theta))^{n-k}\prod_{j=1}^{k}\{\cos(\pi-\theta)+\mu_j\sin(\pi-\theta-\theta_0)\}$. Namely, the absolute value of each factor in $\prod_{j=1}^{k}\{\cos\theta+\mu_j\sin(\theta-\theta_0)\}$ might be considered. The claim will be proved, if the following inequalities are valid:

$$-\cos\theta-\mu\sin(\theta-\theta_0)\leqq\cos(\pi-\theta)+\mu\sin(\pi-\theta-\theta_0)\leqq\cos\theta+\mu\sin(\theta-\theta_0).$$

The first inequality is trivial and for the second one it is reduced to

$$\begin{aligned}&\{\cos\theta+\mu\sin(\theta-\theta_0)\}-\{\cos(\pi-\theta)+\mu\sin(\pi-\theta-\theta_0)\}\\&=2\cos\theta(1-\mu\sin\theta_0).\end{aligned}$$

Since the matrix $cI+sI(\mu_1,\cdots,\mu_k,0,\cdots,0)$ is positive semi-definite, its eigenvalues $c+s\mu_j$ are all of non-negative and so are $\cos\theta+\mu_j\sin(\theta-\theta_0)$ for $j=1,\cdots,k$. Hence, as θ tends to 0, we have

$$\lim_{\theta\to 0}\{\cos\theta+\mu_j\sin(\theta-\theta_0)\}=1-\mu_j\sin\theta_0\geqq 0,$$

which implies that the right inequality is valid. Thus the proof of Lemma 2.2 is complete.

3. Curvature Operator

In this section, the concept of the curvature operator in a Riemannian manifold (M,g) will be introduced and the manifold structures of M which are influenced by some conditions of the operator are investigated. For a point x in M let R_x be an associated curvature operator. A linear map $\rho_x{}^*$ of Λ^2M_x into $\Lambda^2M_x{}^*$ for any point x in M is defined by $u\wedge v\to R_x(.,.,u,v)$ and by this duality an endomorphism ρ_x of $\Lambda^2M_x{}^*$ into itself is manufactured. It turns out that ρ_x satisfies

$$(3.1)\quad \langle\rho_x(u^*\wedge v^*),\ w^*\wedge z^*)\rangle=\langle\rho_x^*(u\wedge v),\ w^*\wedge z^*)\rangle=R_x(w, z, u, v)$$

for any vectors u, v, w and z in M_x, where u^* denotes the dual form in M_x^* associated with the vector u. The operator ρ_x is called a *curvature operator* at x. Since ρ_x is the symmetric operator, each eigenvalue of it is real. If all eigenvalues of ρ_x are contained in the closed interval $[\lambda, \Lambda]$, then one says $\lambda\leqq\rho_x\leqq\Lambda$, and if for any point x in M this property is satisfied, then $\rho(M)$ is said to satisfy the condition $\lambda\leqq\rho(M)\leqq\Lambda$, where $\rho(M)$ is a set which consists of all curvature operators at all points in M.

Now, for an orthonormal basis $\{u_1, \cdots, u_n\}$ of M_x and its dual basis $\{\omega^1, \cdots, \omega^n\}$ for M_x^* relative to $\{u_1, \cdots, u_n\}$, the following equation is given:

$$(3.2)\quad \langle\rho_x(\omega^i\wedge\omega^j),\ \omega^i\wedge\omega^j\rangle=R(u_i, u_j, u_i, u_j)=-g(R(u_i, u_j)u_i, u_j),$$

from which

$$(3.3)\quad \langle\rho_x(\omega^i\wedge\omega^j),\ \omega^i\wedge\omega^j\rangle=K(u_i, u_j),$$

where $K(u_i, u_j)$ means a sectional curvature of a plane section spanned by the orthonormal vectors u_i and u_j. It follows that $K(M)\geqq 0$ if $\rho(M)\geqq 0$. Under the pinching of the curvature operator $\rho(M)$, the curvature tensor R and the Ricci tensor S are also pinched as follows:

$$(3.4)\quad \begin{aligned}&\lambda(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})\leqq -g(R(u_i, u_j)u_k, u_l)\leqq\Lambda(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})\\ &\lambda(n-1)\delta_{ij}\leqq S(u_i, u_j)\leqq\Lambda(n-1)\delta_{ij}.\end{aligned}$$

Thus, if $\lambda\leqq\rho(M)\leqq\Lambda$ then $\lambda\leqq K(M)\leqq\Lambda$. Remark here that the converse is not necessarilly true.

Now, it plays an important role to restrict with the manifold structures of M that the curvature operator $\rho(M)$ is pinched. This is first studied by Yano and Bochner [16]. Suppose that $\lambda\leqq\rho(M)\leqq\Lambda$. Given any p-form ω in $\Lambda^p M^*$, we put

$$(3.5)\quad \begin{aligned}F(\omega)=&\sum_{i,j}\sum_{i_2,\ldots,i_p}S(i, j)\omega(j, i_2, \cdots, i_p)\\ &-\frac{p-1}{2}\sum_{i,j,k,l}\sum_{i_3,\ldots,i_p}R(i, j, k, l)\omega(i, j, i_3, \cdots, i_p)\\ &\omega(k, l, i_3, \cdots, i_p)\end{aligned}$$

then the function $F(\omega)$ can be bounded from below. Namely, it follows from (3. 4) that

$$F(\omega)\geqq\{(n-1)\lambda-(p-1)\Lambda\}|\omega|^2.$$

This implies $F(\omega)>0$ if $\lambda=\Lambda/2$ and $2p<n+1$.

In order to generalize the theorem due to Yano and Bochner, the other expression of the function F will be considered, by making use of the curvature operature. Since components of any p-form ω in Λ^pM^* with respect to the orthonormal basis $\{u_1,\cdots,u_n\}$ for M_x is given by $\omega(i_1,\cdots,i_p)$, where $\{\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}\}$ $(i_1,\cdots,i_p\in\{1,\cdots,n\})$ is an orthonormal basis of Λ^pM^*, the p-form ω is expressed as

$$\omega=\sum_{i_1,\cdots,i_p}\omega(i_1,\cdots,i_p)\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}.$$

For a p-form ω at x we shall consider a family of exterior 2-forms $(i_1,\cdots,i_p)^\omega$ corresponded to the p-form ω, which is defined by

$$(3.6)\qquad (i_1,\cdots,i_p)^\omega=\sum_{k=1}^{p}\sum_{j_k=1}^{n}\omega(i_1,\cdots,i_{k-1},j_k,i_{k+1},\cdots,i_p)\omega^{j_k}\wedge\omega^{i_k}.$$

Moreover a family of scalars $(i_1,\cdots,i_p)^{\theta(\omega)}$ associated with the form ω is produced. The scalar is also defined by

$$(3.7)\qquad (i_1,\cdots,i_p)^{\theta(\omega)}=\langle\rho_x(i_1,\cdots,i_p)^\omega,\ (i_1,\cdots,i_p)^\omega\rangle.$$

we have by (2. 5)

$$(3.8)\qquad F(\omega)=A(\omega)-\frac{p-1}{2}B(\omega),$$

where

$$A(\omega)=\sum_{i,j}\sum_{i_2,\cdots,i_p}S(i,j)\omega(i,i_2,\cdots,i_p)\omega(j,i_2,\cdots,i_p)$$
$$B(\omega)=\sum_{i,j,k,l}\sum_{i_3,\cdots,i_p}R(i,j,k,l)\omega(i,j,i_3,\cdots,i_p)\omega(k,l,i_3,\cdots,i_p)$$

The following Lemma 3. 1 and lemma 3. 2 are due to Meyer [12].

LEMMA 3. 1. $F(\omega)=\dfrac{1}{p}\sum_{i_1,\cdots,i_p}(i_1,\cdots,i_p)^{\theta(\omega)}$

LEMMA 3. 2. *If ω is an exterior p-form on M which does not vanishes* *at x for $1 \leqq p \leqq n-1$, then the associated 2-form is not equal to zero at x.*

By making use of Lemmas 3.1 and 3.2, the following property is varified.

LEMMA 3.3. *Let M be an n-dimensional compact and oriented Riemannian manifold. If all curvature operators satisfy $\rho(M) \geqq 0$ and if there exists a point x_0 in M at which the curvature operator ρ_{x_0} is positive, then M is a real homology sphere.*

Proof. The hypothesis $\rho(M) \geqq 0$ implies that for a point x all eigenvalues of the curvature operator ρ_x are non-negative, so by (3.6) any exterior p-form satisfies the condition:

$$(i_1, \cdots, i_p)^{\theta(\omega)} \geqq 0$$

for any indices $i_1, \cdots, i_p$ in $\{1, 2, \cdots, n\}$. It follows from Lemma 3.1 that $F(\omega) \geqq 0$ and the equality holds true if and only if all scalars $(i_1, \cdots, i_p)^{\theta(\omega)}$ for any indices $i_1, \cdots, i_p$ are equal to zero. It implies that in the equation

$$(\Delta\omega, \omega) = \| \nabla\omega \|^2 + Q(\omega),$$

where $Q(\omega) = \int_M F(\omega) dV_M$, $\Delta\omega$ is the Laplacian of ω and $\nabla\omega$ is the covariant derivative of ω, the second term $Q(\omega)$ of the right hand side is non-negative. If the p-form ω is harmonic, then $\Delta\omega = 0$ and we obtain that $F(\omega)$ and $\nabla\omega$ vanish everywhere on M. Thus the p-form ω is parallel.

On the other hand, since the curvature operator ρ_{x_0} at the given point x_0 is positive by means of the assumption, the scalar $(i_1, \cdots, i_p)^{\theta(\omega)x_0}$ is non-negative, and it is equal to zero if and only if the associated 2-form of ω vanishes at x_0. However, since $F(\omega)$ vanishes everywhere, Lemma 3.1 yields all scalars $(i_1, \cdots, i_p)\theta(\omega)$ are equal to zero for any indices $i_1, \cdots, i_p$ and therefore

$$(i_1, \cdots, i_p)^{\omega_{x_0}} = 0.$$

It follows from Lemma 3.2 that the p-form ω is equal to zero at the point x_0. Since ω is parallel, the norm $\| \omega \|$ vanishes everywhere on M. Therefore, the theorem due to Hodge asserts $H^p(M, R) = 0$ for $0 < p < n$. This completes the proof.

THEOREM 3.4. *Let M be an $n(\geqq 3)$-dimensional compact and oriented*

Riemannian manifold where all sectional curvatures are greater than and equal to a constant c, and $\widetilde{M}$ *be an* $(n+2)$*-dimensional complete simply connected Riemannian manifold of constant curvature c. If M is isometrically immersed in* $\widetilde{M}$ *and if* $c>0$ *or* $c=0$ *and there exists a point* x_0 *at which all sectional curvatures are positive, then M is a real homology sphere.*

Proof. Let f be an isometric immersion of M into $\widetilde{M}$. For any point x in M we shall denote $f(x)$ in $\widetilde{M}$ by the same symbol x since there is no danger of confusion and moreover since the computation is local. Furthermore, a tangent vector u at x is identified with the tangent vector $df_x(u)$. Thus the tangent space M_x is a subspace of the tangent space $\widetilde{M}_x$ to the ambient space at x. Let N_x be the orthogonal complement of M_x in $\widetilde{M}_x$, which is called the normal space to M at x, and σ be the second fundamental form of the immersion f. For the triple (M_x, N_x, σ_x) at each point x in M, algebraic preliminaries which are prepared for in §2 can be applied. Let R_σ be the associated curvature form on M_x which is defined by (2. 1) and K_σ be the real valued map on $M_x\times M_x$ defined by (2. 2). Then it follows from the Gauss equation for the theory of submanifolds in a real space form that we have

$$R_\sigma(u\wedge v,\ w\wedge z)=R(u,v,w,z)-c(\langle u,w\rangle\langle v,z\rangle-\langle u,z\rangle\langle v,w\rangle)$$

for any vectors u, v, w and z in M_x, and hence

$$K_\sigma(u,v)=R_\sigma(u\wedge v,\ u\wedge v)=(K(u,v)-c)(\|u\|^2\|v\|^2-\langle u,v\rangle^2).$$

By the assumption of the theorem it means $K_\sigma(u,v)\geqq 0$. Thus, by taking Lemma 2. 1 the associated form R_σ satisfies $R_\sigma\geqq 0$. Hence, the curvature operator ρ_x at x satisfies $\rho_x\geqq c$, because its operator ρ_x is given by $\langle\rho_x(u^*\wedge v^*),\ w^*\wedge z^*\rangle=R_x(u,v,w,z)$. On the other hand, the assumption shows that $\rho(M)\geqq c\geqq 0$ and there exists a point x_0 at which $\rho_{x_0}>0$. It implies that theorem 3. 4 can be applied. This concludes the proof.

4. Proof of Main theorem

This section is devoted to determining completely the k-th integral homology group of M. Let M be an n-dimensional compact oriented Riemannian manifold. For a given function h on M, a point x in M is called a critical point of h if $(dh)_x=0$ and a critical form of h is said to be *non-degenerate* if the hessian form of h is non-degenerate. In this case the dimension of maximal dimensional subspaces of the tangent space M_x on which the hessian form is negative definite is called the *index of the critical point* x. Set

$$\beta_k(h)=\text{the number of critical points of index } k \text{ of } h,$$
$$\beta_k(M)=\text{Min}\{\beta_k(h)\,;\ h \text{ is a Morse function}\},$$

where a Morse function means a function on M with only non-degenerate critical points.

For arbitrary field F, let $b_k(M, F)$ be the rank of $H^k(M, F)$ as a module over F, which is called a k-th Betti number. If F is a field of characteristic zero, these modules are vector space over F, so $b_k(M, F)$ is the dimension of the vector space $H^k(M, F)$. Since M is compact and oriented, by Poincaré duality theorem

$$b_k(M, F)=b_{n-k}(M, F).$$

On the other hand, it follows from Morse's inequalities [13] that one yields

$$b_k(M, F)\leqq\beta_k(M)\leqq\beta_k(h), \tag{4.1}$$

$$\sum_{i=1}^{k}(-1)^i b_{k-i}(M, F)\leqq\sum_{i=1}^{k}(-1)^i\beta_{k-i}(h). \tag{4.2}$$

Let M be now isometrically immersed in R^N under the isometric immersion f. Let B be the bundle of unit normal spheres over M, S^{N-1} be a unit sphere of R^N centered with origin, and G be a Gauss map of B into S^{N-1}. A Gauss map is a map which assigns to each normal in B a unit vector through the origin of R^N parallel to the normal. But the image of ξ under the Gauss map may be identified with itself. For any ξ in S^{N-1} a height function h on M is defined by

$$h_\xi(x)=\langle f(x),\ \xi\rangle$$

where $\langle\,,\rangle$ means a Euclidean product on R^N.

On the other hand, for the isometric immersion f of M into R^N and a normal ξ in B, let A_ξ be the symmetric transformation associated with the second fundamental form on M induced by f. Then the total curvature $\tau(M)$ of M immersed in R^N by the isometric immersion f is defined by

(4.3) $$\tau(M)=\tau(M,f,R^N)=\frac{1}{\text{vol } S^{N-1}}\int_B |\det A_\xi|\, dV_B,$$

where dV_B means the volume element of B and vol S^{N-1} denotes the volume of unit shere S^{N-1} in R^N. By means of the work of Chern and Lashof [4, 5] the total curvature $\tau(M)$ of M is the volume of the image of B under the parallelism G. The critical point ξ of the map, that is, the point where the functional determinant of the map is zero, is exactly the point where the hessian of the height function h_ξ is of rank $<n$. It is well known that x is a critical point of h_ξ if and only if ξ is normal at x, and at a critical point the hessian of h_ξ is the second fundamental form of M in the direction ξ. Expresses somewhat differently, the critical point ξ of G is the normal to M where the height function h_ξ has degenerate critical points. By Sard's theorem, their image on S^{N-1} has measure zero. Thus, for almost all unit normals ξ the height function h_ξ on M, with fixed ξ, has only non-degenerate critical points, it is the Morse function.

Choose a normal ξ in B with respect to which the height function h_ξ is a Morse function. Following Kuiper [9], the total curvature τ_k of index k is defined by

(4.4) $$\tau_k=\frac{1}{\text{vol } S^{N-1}}\int_{S^{N-1}} \beta_k(\xi)\, dV_S,$$

where $\beta_k(\xi)=\beta_k(h_\xi)$. This can be regarded as the average number of critical points of index k. When the normal ξ is replaced by $-\xi$, critical points of index k change into critical points of index $n-k$ and it means

(4.5) $$\tau_{n-k}=\tau_k.$$

Now, for the Morse function h it follows from (4.1) and (4.4) that one yields $b_k(M,F)\leqq\beta_k(\xi)$. By averaging this inequality over S^{N-1}, the following relation is obtained: $b_k(M,F)\leqq\tau_k$. By Morse's inequalities,

$$b_1(M,F)-b_0(M,F)\leqq\beta_1(\xi)-\beta_0(\xi).$$

Again, by integralizing this inequality, the following relation of the Betti number to the total curvatures is given:

$$b_1(M,F)-b_0(M,F)\leqq\tau_1-\tau_0.$$

Hence $b_1(M,F)\leqq\tau_1-\tau_0+1$, because M is connected. Since M is compact and oriented, the equation (4. 5) and the Poincaré duality theorem imply

$$b_{n-1}(M, F) \leqq \tau_{n-1} - \tau_n + 1,$$

which will give the following inequality:

$$(4.6) \qquad b_1(M, F) + \cdots + b_{n-1}(M, F) \leqq \tau_1 + \cdots + \tau_{n-1} - \tau_0 - \tau_n + 2,$$

for any field F.

Since the image of the bundle B of unit normal spheres over M under the map G is the same as the set of points ξ in S^{N-1}, each counted a number of times equal to the number of critical points of the height function h_ξ on M, and since the total curvature is the volume of the image of B, it follows from (4. 3) and (4. 4) that we obtain

$$\tau(M) = \tau_0 + \tau_1 + \cdots + \tau_n.$$

Let B_+ be the set of all unit normals ξ such that A_ξ is definite and B_0 be the set of all unit normal ξ such that A_ξ is not definite. Then, by the definition the total curvature $\tau(M)$ is reduced to

$$\tau(M) = \frac{1}{\text{vol } S^{N-1}} \left(\int_{B_+} |\det A_\xi| dV_B + \int_{B_0} |\det A_\xi| dV_B \right).$$

and the meaning of the hessian of the height function implies

$$\tau_1 + \cdots + \tau_{n-1} \leqq \frac{1}{\text{vol } S^{N-1}} \int_{B_0} |\det A_\xi| dV_B.$$

Combining together with some results obtained above, we have

$$(4.7) \qquad \tau_1 + \cdots + \tau_{n-1} - \tau_0 - \tau_n \leqq \frac{1}{\text{vol } S^{N-1}} \left(\int_{B_0} |\det A_\xi| dV_B - \int_{B_+} |\det A_\xi| dV_B \right).$$

For the remainder of this section we are concerned with the proof of Main theorem. Given a point x in M, the tangent space M_x and the normal space N_x with respect to the Euclidean scalar product in R^{n+2} can be regarded as vector spaces V and W respectively, and as a symmetric map of V into W the second fundamental form σ_x at x is considered. For this triple $\{M_x, N_x, \sigma_x\}$, one can select an orthonormal basis $\{\xi_{n+1}, \xi_{n+2}\}$ for W such that $\langle \sigma_x(u, u), \xi_\alpha \rangle \geqq 0$ for any index α and any vector

u in M_x, and for this basis

$$|\det A_{\pi-\theta}| \leqq |\det A_\theta| \qquad \text{for } \theta \in [0, \pi/2],$$

where $A_\theta = A_{\xi(\theta)}$, $\xi(\theta) = \cos\theta \cdot \xi_{n+1} + \sin\theta \cdot \xi_{n+2}$. Hence

$$\int_{B_0} |\det A_\xi| dV_B \leqq \int_{B+} |\det A_\xi| dV_B,$$

because an eigenvalue of the transformation A_ξ is non-negative for $\theta \in [0, \pi/2]$. Since there exists a point x_0 at which all sectional curvatures are positive by the assumption, we may suppose that all sectional curvatures are positive on a compact set C with positive measure, in which the point x_0 is contained, because sectional curvatures are continuous on M, Accordingly

$$|\det A_{\pi-\theta}| < |\det A_\theta| \qquad \text{on } C.$$

Thus we have

$$\int_{B_0} |\det A_\xi| dV_B < \int_{B+} |\det A_\xi| dV_B,$$

which is an inequality due to Chen [3]. Combining (4.6) and (4.7) together with the above inequality, one obtains

$$b_1(M, F) + \cdots + b_{n-1}(M, F) < 2, \tag{4.8}$$

for any field F.

First of all, under the hypothesis of Main theorem it will be shown that M must be simply connected. Suppose that M is not simply connected. The fundamental group $\pi_1(M)$ contains a subgroup isomorphic to Z_p for some prime number p. Let $(\widetilde{M}, \pi)$ be a Riemannian covering of M corresponding to this subgroup, where π is the covering projection. It is easily seen that $\widetilde{M}$ is also compact and oriented over Z_p, and $\widetilde{M}$ satisfies the same property for the curvature as that of M. Moreover the composition $f \circ \pi$ of the projection π with the isometric immersion f becomes again an isometric immersion of $\widetilde{M}$ into R^{n+2}. For the Betti number $b_k(\widetilde{M}, Z_p) = \dim H^k(\widetilde{M}, Z_p)$, the Riemannian covering $(\widetilde{M}, \pi)$ gives $b_1(\widetilde{M}, Z_p) = b_{n-1}(\widetilde{M}, Z_p) = 1$. Under this situation $(\widetilde{M}, f \circ \pi, R^{n+2})$ the inequality (4.8) can be obtained, and these properties contradict each other. Thus M is simply connected.

Now, since Theorem 3. 4 means that M is a real homology sphere, we need only show that M has no torsion. In general, the k-th homology group $H^k(M, K)$ over Z is expressed as

$$H^k(M, Z) = \underbrace{Z\oplus\cdots\oplus Z}_{b_k}\oplus Z_{t_k{}^1}\oplus\cdots\oplus Z_{t_k{}^{c_k}},$$

Then the torsion coefficient $\{t_k{}^j\}$ satisfy the property: $t_k{}^j | t_k{}^{j+1}$, by means of the Poincaré duality theorem and the universal coefficient theorem it is easily seen that $t_k{}^j = t_{n-k-1}{}^j$ for any $j=1, \cdots, c_k = c_{n-k-1}$. On the other hand, in this case the k-th homology group $H^k(M, R)$ satisfies $H^k(M, R) = R\oplus\cdots\oplus R$ (b_k-times). On the contrary, for a prime number p, it is well known that the universal coefficient theorem implies again

$$H^k(M, Z_p) = Z_p\oplus\cdots\oplus Z_p (b_k + \lambda_k + \lambda_{k-1}\text{-times})$$

where λ_k is the number of integers j such that p is a divisor of the integer $t_k{}^j$.

By coming back to the proof and again by making use of Theorem 2. 2, one finds

$$H^k(M, R) = H^{n-k}(M, R) = 0 \qquad \text{for any } k=1, \cdots, n-1.$$

Hence, under this situation, $b_k = 0$ for any $k=1, \cdots, n-1$. Now, suppose that M contains torsions, that is, $H^k(M, Z)$ has the torsion part, then there exists a prime p such that $\lambda_k > 0$ and therefore

$$b_k(M, Z_p) = b_{n-k}(M, Z_p) \geqq 1.$$

If k is different from $n-k$, then it contradicts to the inequality (4. 8) accordingly $k = n-k$. Thus, if M is not a homology sphere over the integers, it must be even dimensional, say $n=2m$, and all its torsions must lie in $H^m(M, Z)$. Hence we get $t_m{}^j = t_{n-m-1}{}^j = t_{m-1}{}^j$, because of $n = 2m$, which implies

$$\text{torsion } H^m(M, Z) = \text{torsion} H^{m-1}(M, Z).$$

Again, by Poincaré duality theorem, it contradicts to $k=m$. Thus we have $H^k(M, Z) = 0$ for $k = 1, 2, \cdots, n-1$. It finishes the proof of Main theorem.

Acknowledgement The authers wish to express his gratitude to Professor

Hisao Nakagawa for his advices and encouragement.

Bibliography

1. Y.Y. Baldin and F. Mercuri, *Isometric immersions in codimension two with non-negative curvature*, Math. Z., 173 (1980), 111-117.
2. R.I. Bishop, *The holonomy algebra of immersed manifolds of codimension two*, J. Differential Geometry, 2 (1968), 347-353.
3. C.S. Chen, *On tight isometric immersion of codimension two*, Amer. J. Math., 94 (1972), 974-990.
4. S.S. Chern and R.K. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math., 79 (1957), 306-318.
5. S.S. Chern and R.K. Lashof, *On the total curvature of immersed manifolds II*, Michigan Math. J., 5 (1958), 5-12.
6. S. Gallot and D. Meyer, *Operateur de courbure et Laplacien des forme differentielle d'une variétés riemannienne*, J. Math. pures et appl., 54 (1975), 259-284.
7. S.I. Goldberg, *Curvature and homology*, Academic Press, 1962.
8. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I, II*, Interscience Publishers, 1963, 1969.
9. N.H. Kuiper, *Minimal total absolute curvature for immersions*, Invent Math., 10 (1970), 209-238.
10. A. Lichnerowicz, *Géométrie des groupes de transformations*, Ravaux et Recherches Math., Qunod, Paris, 1958.
11. S. Maclane, *Homology*, Academic Press, 1963.
12. D. Meyer, *Sur les variétés riemanniennes à operateur de courbure positif*, C.R. Acad. Sc. Paris, 272 (1971), 482-485.
13. J. Milnor, *Morse Theory*, Ann. of Math. Studies, No. 51, Princeton Univ. Press, 1963.
14. J.D. Moore, *Codimension two submanifolds of positive curvature*, Proc. Amer. Math. Soc., 70 (1978), 72-74.
15. A. Weinstein, *Positively curved n-manifolds in R^{n+2}*, J. Differential Geometry, 4 (1970), 1-4.
16. K. Yano and S. Bochner, *Curvature and Betti Number*, Annals of Math. Studies, No. 32, Princeton Univ. Press, 1953.

Hyosung Women's University
Taegu 634, Korea

Kyung Nam University
Masan 610, Korea