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THE UNIQUENESS OF THE CAUCHY PROBLEM FOR THE SECOND ORDER ELLIPTIC OPERATORS

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One of the conditions used to prove the Calderon's theorem was the smoothness of the characteristic roots. However in some cases the smoothness condition can be relaxed.

When Zuily [1, 2] proved the uniqueness of the Cauchy problem for the second order elliptic differential operator in R^2 with analytic principal part and bounded lower order terms, he reduced the problem to the second order elliptic operator with C^∞ coefficients. In the proof of the reduction he considered the following two cases separately;

Case 1. $t\rightarrow\Delta(0,t)$ has a zero of finite order

Case 2. $t\rightarrow\Delta(0,t)$ vanishes identically
but $\Delta(x,t)$ does not vanish identically.

In case 1 we have a easy proof and in case 2 the uniqueness follows from the Calderons theorem and the reductio ad absurdum method. In this paper we treat the above two cases at a single stroke by reducing the case 2 to case 1.

THEOREM 1. *Let P be a second order elliptic differential operators in a neighborhood V of the origin $0\in R^2$ with bounded lower order terms such that the discriminant Δ of P has a zero of finite order at $0\in R^2$. Let $S=\{x\in V;\ \phi(x)=\phi(0)\}$ be a C^2 non-characteristic hypersurface near 0. Then there exists a neighborhood W of the origin such that every $u\in C^\infty(V)$ satisfying*

$$\begin{cases} Pu=0 \text{ in } V \\ u=0 \text{ in } \{x\in V;\ \phi(x)\le\phi(0)\}\end{cases}$$

vanishes in W.

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LEMMA 1. *Let* $P=aD_t^2+2bD_xD_t+cD_x^2+\alpha D_x+\beta D_t+\gamma$ *be a second order differential operator in a neighborhood of the origin with* $a, b, c\in C^\infty$ *and* $\alpha, \beta, \gamma\in L^\infty$ *and let* κ *be a coordinate transformation near* 0 *with* $\kappa(0)=0$. *Define* P^κ *by* $P^\kappa u=P(u\circ\kappa)$. *Then the discriminant of* P^κ *takes the following form:*

$$\Delta^\kappa=[\Delta(\det\kappa')^2]\circ\kappa^{-1}.$$

Proof. Let $\kappa(x,t)=(\kappa_1(x,t),\ \kappa_2(x,t))$. Then $P^\kappa(y,s,D)=\tilde{a}D_s^2+2\tilde{b}D_yD_s+\tilde{c}D_y^2+\tilde{\alpha}D_y+\tilde{\beta}D_s+\tilde{\gamma}$. Here

$$\tilde{a}=\left[a\left(\frac{\partial\kappa_2}{\partial t}\right)^2+2b\frac{\partial\kappa_2}{\partial t}\ \frac{\partial\kappa_2}{\partial x}+c\left(\frac{\partial\kappa_2}{\partial x}\right)^2\right]\circ\kappa^{-1}$$

$$\tilde{b}=\left[a\frac{\partial\kappa_2}{\partial t}\ \frac{\partial\kappa_1}{\partial t}+b\left(\frac{\partial\kappa_1}{\partial t}\ \frac{\partial\kappa_2}{\partial x}+\frac{\partial\kappa_1}{\partial x}\ \frac{\partial\kappa_2}{\partial t}\right)+c\frac{\partial\kappa_2}{\partial x}\ \frac{\partial\kappa_1}{\partial x}\right]\circ\kappa^{-1}$$

$$\tilde{c}=\left[a\left(\frac{\partial\kappa_1}{\partial t}\right)^2+2b\frac{\partial\kappa_1}{\partial x}\ \frac{\partial\kappa_1}{\partial t}+c\left(\frac{\partial\kappa_1}{\partial x}\right)^2\right]\circ\kappa^{-1}.$$

Direct calculation shows that $\Delta^\kappa=\tilde{b}^2-\tilde{a}\tilde{c}=[\Delta(\det\kappa')^2]\circ\kappa^{-1}$.

LEMMA 2. *Let* f *be a smooth function in a neighborhood* V *of the origin in* R^2 *which has a zero of finite order at* 0. *Then there exists a coordinate transformation* κ *such that* $D_1^kf(0)\neq 0$ *for some positive integer* k.

Proof. By assumption $f(x,t)=\sum_{|\alpha|=k}C_\alpha(x,t)^\alpha+O(|(x,t)|^{k+1})$, where

$$C_\alpha=\frac{\partial^\alpha f(0)}{\alpha!}.$$

If $\dfrac{\partial^k f}{\partial x^k}(0)\neq 0$, we are done. Otherwise let $\kappa(x,t)=(-\lambda t+x,t)$.

Then

$$\begin{aligned}(f\circ\kappa^{-1})(y,s)&=f(\lambda s+y,s)\\&=\sum_{|\alpha|=k}C_\alpha(\lambda s+y,s)^\alpha+(|(y,s)|^{k+1}).\\&=(\sum_{|\alpha|=k}C_\alpha\lambda^{\alpha_1})s^k+\sim.\end{aligned}$$

Hence it suffices to choose λ so that $\sum_{|\alpha|=k}C_\alpha\lambda^{\alpha_1}\neq 0$, which is clearly possible.

Sketch of the proof of Theorem 1. By Lemma 1 and Lemma 2 we may assume that $S=\{t=0\}$, i.e., $\phi(x,t)=t$ and that $t\to\Delta(0,t)$ has a zero of finite order at 0. Then the rest of the proof is almost classical (See Zuily [1]).

REMARK. If P is a second order elliptic differential operator then P satisfies the hypotheses of Theorem 1.

References

1. C. Zuily: *Uniqueness and Non-Uniqueness in the Cauchy Problem.* Progress in Mathematics Vol. 33, Birkhäuser (1983).
2. C. Zuily: *Unicité du Probleme de Cauchy pour des operateurs elliptiques a caracteristiques de Hautes Multiplicites.* Prepublication 84-T-26, Université de Paris-Sud.