J. Korean Math. Soc. **23**(1986), No. 2, pp. 223-239

# SUBMANIFOLDS OF KAEHLERIAN MANIFOLDS

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## 0. Introduction

The research of submanifolds in Kaehlerian manifolds is a wide and interesting branch of differential geometry, and many geometers have concerned themselves with the study of structures induced on submanifolds and geometric properties of $CR$-submanifolds (see [1, 2, 4, 5, 7, 9, 11, 15, 17] and [18]). In order to investigate these submanifolds from an integrated view-point, Y. Tashiro and one of the present authors ([15]) introduced the notion of metric compound structures on a Riemannian manifold.

In the present paper, we see that the above mentioned structures or submanifolds are characterized by the rank $r$ of a map $v$ and some scalar fields associated with the rank. The $r$-plane section on submanifolds are certain subbundles of normal bundles and related to the rank $r$. The main purpose of the present paper is to investigate geometric structures of a submanifold with concurrent and umbilical $r$-plane sections in Kaehlerian manifolds. Conditions for such a submanifold to be conformal to a warped product, a Euclidean space or a sphere, and to be isometric to a sphere or a Sasakian manifold will be obtained.

In Paragraph 1, we shall define some scalar fields associated with the rank of a map $v$ and characterize the notion of $CR$-submanifolds of almost Hermitian manifolds in terms of the rank and scalar fields. In Paragraph 2, we shall discuss $r$-plane sections on a submanifold $M$ of Kaehlerian manifolds. After a brief survey of the mean curvature vector field of $M$ in Paragraph 3, we shall give geometric structures of the submanifold with a concurrent $r$-plane section in Paragraph 4. Paragraph 5 will be devoted to research of properties of the submanifold $M$ with an umbilical $r$-plane section.

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Received May 31, 1986.
This work was supported by a grant from the Korea Science and Engineering Foundation, 1984-86.

Throughout this paper we assume that manifolds and quantities are differentiable of class $C^\infty$. Unless otherwise stated, indices run over the following ranges

$$A, B, C, D, \ldots = 1, 2, 3, \ldots\ldots\ldots, m,$$
$$h, i, j, k, \ldots = 1, 2, \ldots, n,$$
$$p, q, r, s, \ldots = \qquad n+1,\ n+2, \ldots, m,$$
$$a, b, c, d, \ldots = 1, 2, \ldots, r$$

respectively and summation convention is applied to repeated indices on their own ranges.

## 1. Constant rank of a map $v$

Let $\tilde{M}$ be an $m$-dimensional almost Hermitian manifold with the structure $(G, J)$, where $G$ is the almost Hermitian metric tensor and $J$ the almost complex structure of $\tilde{M}$. The structure $(G, J)$ satisfies the relation

$$J^2 = -I,$$

I being the identity tensor field of $\tilde{M}$, and

$$G(J\tilde{X}, J\tilde{Y}) = G(\tilde{X}, \tilde{Y}) \tag{1.1}$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on $\tilde{M}$.

Let $M$ be an $n$-dimensional differentiable manifold and $i$ immersion of $M$ into $\tilde{M}$. In terms of local coordinates $(x^h)$ of $M$ and $(y^A)$ of $\tilde{M}$, the immersion $i$ is locally expressed by the parametric equations

$$y^A = y^A(x^h).$$

If we put

$$B_i{}^A = \partial_i y^A, \quad \partial_i = \partial/\partial x^i,$$

then $B_i = (B_i{}^A)$ are $n$ local vector fields on $M$ spanning the tangent space $T_x(M)$ at every point $x$ of $M$. A Riemannian metric tensor $g = (g_{ji})$ of $M$ is naturally induced from $G$ of $\tilde{M}$ as

$$g_{ji} = G(B_j, B_i).$$

We can choose $m-n$ mutually orthogonal unit normal vector fields $C_p = (C_p{}^A)$ to $M$, Then the vector fields $B_i$ and $C_p$ span the tangent space $T_x(\tilde{M})$ of $\tilde{M}$ at every point $x$ of $M$ and the matrix $B$ defined by

$$B = (B_j, C_p)$$

is regular. We have

$$ {}^tBGB=\begin{pmatrix} g_{ji} & 0 \\ 0 & \delta_{qp} \end{pmatrix}, $$

and $\delta_{qp}=G(C_q, C_p)$ form the induced metric of the normal space $T_x^{\perp}(M)$ of $M$ at each point $x$ of $M$.

If we put

$$ B^{-1}JB=\begin{pmatrix} f_i{}^h & -v_q{}^h \\ v_{pi} & f_{qp} \end{pmatrix} $$

then the map $f=(f_i{}^h)$ is an endomorphism of the tangent bundle $T(M)$ of $M$ and $f^{\perp}=(f_{qp})$ is that of the normal bundle $T^{\perp}(M)$ of $M$. The matrix $v=(v_q{}^h)$ is a map of $T^{\perp}(M)$ into $T(M)$, that is, $v_q{}^h n_q$ for any normal vector field $N=n_pC_p$ to $M$ are tangent components of $JN$. Since components of the almost complex structure $J$ are skew-symmetric, so are the components $f_{ji}=G(JB_j, B_i)$ of $f$ and $f_{qp}=G(JC_q, C_p)$ of $f^{\perp}$.

The transforms of the tangent vectors $B_i$ and the normal vectors $C_p$ of $M$ by $J$ are expressed in the form

(1.2) $$ JB_i=f_i{}^hB_h+v_{pi}C_p, $$

(1.3) $$ JC_q=-v_q{}^hB_h+f_{qp}C_p, $$

where $v_{pi}=v_p{}^hg_{ih}$. Applying $J$ to (1.2) and (1.3), we have the relations

(1.4) $$ f_j{}^if_i{}^h=-\delta_j^h+v_{pj}v_p{}^h, $$

(1.5) $$ f_j{}^iv_{pi}=-v_{qj}f_{qp}, \qquad v_q{}^if_i{}^h=-f_{qp}v_p{}^h, $$

(1.6) $$ f_{rq}f_{qp}=-\delta_{rp}+v_r{}^iv_{pi}, $$

where $\delta_j^h$ and $\delta_q^p$ are components of the identity I. The relation (1.1) is equivalent to

(1.7) $$ g_{kh}f_j{}^kf_i{}^h=g_{ji}-v_{pj}v_{pi}. $$

Now we assume that the rank of the map $v: T^{\perp}(M) \longrightarrow T(M)$ is equal to a constant $r(0\leq r\leq \min\{n, m-n\})$ almost everywhere on $M$. Then there exist linearly independent vector fields $V_a=V_{(a)}{}^hB_h$ on $M$ and $N_a=n_{(a)q}C_q$ normal to $M$ such that

(1.8) $$ v_q{}^h=n_{(a)q}V_{(a)}{}^h. $$

Moreover we may normalize the vector fields $N_a$ such as

$$ G(N_b, N_a)=\delta_{ba}. $$

If we put

(1.9) $$ \lambda_{ba}=G(JN_b, N_a), $$

then these are $r(r-1)/2$ scalar fields on $M$. From the relations (1.4) to (1.9), we have

(1.10) $f^2X=-X+v_a(X)V_a$,

(1.11) $fV_a=-\lambda_{ab}V_b$, $v_a(fX)=\lambda_{ab}v_b(X)$,

(1.12) $f^\perp N_a=\lambda_{ab}N_b$,

(1.13) $g(fX, fY)=g(X, Y)-v_a(X)v_a(Y)$

for any vector fields $X$ and $Y$ on $M$, where $v_a$ is the associated 1-form of $V_a$. Putting $X=V_a$ into (1.10) and using (1.11), we obtain

(1.14) $v_a(V_b)=\delta_{ab}-\lambda_{ac}\lambda_{bc}$.

Therefore the relation (1.6) is reduced to

(1.15) $(f^\perp)^2N=-N+(\delta_{ab}-\lambda_{ac}\lambda_{bc})G(N, N_a)N_b$

for any vector field $N$ normal to $M$. Moreover we see that the transforms (1.2) and (1.3) are reduced to

(1.16) $JX=fX+v_a(X)N_a$

for any vector field $X$ on $M$, or specially

(1.17) $JV_a=-\lambda_{ab}V_b+(\delta_{ab}-\lambda_{ac}\lambda_{bc})N_b$,

and

(1.18) $JN=-G(N, N_a)V_a+f^\perp N$

for any vector field $N$ normal to $M$, or specially

(1.19) $JN_a=-V_a+\lambda_{ab}N_b$.

We define the distributions $D$ and $D_c$ of the tangent space $T_x(M)$, $x\in M$, by

$$D=\text{span }\{V_1, V_2, \dots, V_r\},$$
$$D_c=\{X\in T_x(M)\,|\,g(X, V_a)=0\},$$

which are orthogonal complementary to one another. Then it is easily seen from (1.16) that $D_c$ is a holomorphic distribution for $J$ and of even-dimension. Thus we have

THEOREM 1.1 ([7]). *Let $M$ be a submanifold immersed in almost Hermitian manifolds and $v$ be the map of $T^\perp(M)$ into $T(M)$. Then $M$ is even or odd-dimension according as the rank of $v$ is even or odd.*

A. Bejancu ([1]), D.E. Blair and B.Y. Chen ([2]) have recently introduced the notion of $CR$-submanifolds in Hermitian manifolds, which contains that of holomorphic (or invariant), anti-holomorphic (or anti-invariant) and generic submanifolds (for instance, see [5], [11], [17] and [18] as to these submanifolds). Since the distribution $D_c$ is holomorphic, we can easily verify that $M$ is holomorphic (resp. anti-holomorphic or generic) if and only if $r=0$ (resp. $r=n$ or $r=m-n$).

If $0<r<\min\{n, m-n\}$ and the scalar fields $\lambda_{ab}$ vanish identically, then it follows from (1.17) that the distribution $D$ is anti-holomorphic and hence $M$ is a $CR$-submanifold. Conversely, if $M$ is a $CR$-submanifold, we can define scalar fields $\lambda_{ab}$ by $\lambda_{ab}=G(JE_a, E_b)$ for orthonormal vector fields $E_a$ in the anti-holomorphic distribution of $M$. It is clear that the scalars $\lambda_{ab}$ vanish identically. Thus we can state

THEOREM 1.2. *Let $M$ be an $n$-dimensional submanifold in $m$-dimen sional Hermitian manifolds. Suppose that the rank of the map $v: T^{\perp}(M) \longrightarrow T(M)$ is equal to a constant $r$ and the scalar fields $\lambda_{ab}$ are given by* (1.9). *Then*

(1) *$M$ is a holomorphic submanifold if and only if $r=0$,*
(2) *$M$ is an anti-holomorphic submanifold if and only if $r=n$,*
(3) *$M$ is a generic submanifold if and only if $r=m-n$.*
(4) *$M$ is a $CR$-submanifold if and only if $\lambda_{ab}=0$.*

## 2. $r$-plane sections on submanifolds

In the sequel, we assume that $M$ is a submanifold immersed in a Kaehlerian manifold $\tilde{M}$ and the rank of the map $v: T^{\perp}(M) \longrightarrow T(M)$ is equal to a constant $r$ almost everywhere on $M$. Let $\tilde{\nabla}$ and $\nabla$ be the operators of covariant differentiation with respect to the metric $G$ on $\tilde{M}$ and to the induced metric $g$ on $M$ respectively. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$
$$\tilde{\nabla}_X N = -A_N X + \nabla_X{}^{\perp} N \tag{2.2}$$

for any vector fields $X$ and $Y$ on $M$ and $N$ normal to $M$, where $h$ is the second fundamental form, $\nabla^{\perp}$ the linear connection induced in the normal bundle $T^{\perp}(M)$, called the *normal connection*, and $A_N$ the second fundamental tensor with respect to $N$. The second fundamental form $h$ and tensor $A_N$ are related by

$$G(h(X, Y), N) = g(A_N X, Y). \tag{2.3}$$

Differentiating (1.16) covariantly along $M$ and taking account of (2.1), (2.2) and (2.3), we obtain

$$f\nabla_Y X - g(A_a X, Y)\, V_a + v_a(\nabla_Y X) N_a + f^{\perp} h(X, Y)$$

$=(\nabla_Y f)X+f\nabla_Y X-v_a(X)A_aY+h(fX, Y)+Y(v_a(X))N_a+v_a(X)\nabla_Y^{\perp}N_a,$
where we have put $A_aX=A_{N_a}X$. Taking tangential and normal components of the equation, we have

$$(2.4)\qquad (\nabla_Y f)X=v_a(X)A_aY-g(A_aX, Y)V_a,$$

$$(2.5)\qquad f^{\perp}h(X, Y)=h(fX, Y)+(\nabla_Y v_a)(X)N_a+v_a(X)\nabla_Y^{\perp}N_a.$$

Since, for example, the relation

$$G(f^{\perp}h(X, Y), N_a)=-G(h(X, Y), f^{\perp}N_a)=-\lambda_{ab}g(A_bX, Y)$$

is satisfied by (1.12), it follows from (2.5) that

$$(\nabla_Y v_a)(X)=-\lambda_{ab}g(A_bX, Y)+g(X, fA_aY)-L_{ab}(Y)v_b(X)$$

or equivalently

$$(2.6)\qquad \nabla_X V_a=-\lambda_{ab}A_bX+fA_aX-L_{ab}(X)V_b,$$

where $L_{ab}(X)=G(N_a, \nabla_X^{\perp}N_b)$ is a scalar field on $M$ and skew-sysmmetric in $a$ and $b$. Similarly, differentiating (1.18) covariantly and taking tangential and normal components, we have

$$(2.7)\qquad fA_NX-A_{f^{\perp}N}X-G(N, \nabla_X^{\perp}N_a)V_a$$
$$=G(N, N_a)\ (fA_aX-\lambda_{ab}A_bX-L_{ab}(X)V_b),$$

$$(2.8)\qquad (\nabla_X^{\perp}f^{\perp})N=G(N, N_a)h(X, V_a)-v_a(A_NX)N_a.$$

It follows from (1.9), (1.19) and (2.2) that

$$(2.9)\qquad X\lambda_{ab}=v_a(A_bX)-v_b(A_aX)+\lambda_{ac}L_{cb}(X)-\lambda_{bc}L_{ca}(X).$$

The *mean curvature vector field* $H$ of $M$ in $\tilde{M}$ is defined by

$$H=(1/n)(\text{trace } A_p)C_p.$$

For a unit normal vector field $N$ to $M$, $\tau=(1/n)$trace $A_N$ is called the *mean curvature belonging to* $N$. If the mean curvature vector field $H$ of $M$ vanishes identically, then $M$ is said to be *minimal*. A normal vector field $N$ is said to be an *umbilical* (resp. a *geodesic*) *section* on $M$, or $M$ is called *umbilical* (resp. *geodesic*) *with respect to* $N$, if $A_NX=\tau X$ (resp. $A_NX=0$) for any vector field $X$ on $M$. If $M$ is umbilical (resp. geodesic) with respect to all unit normal vector fields to $M$, then $M$ is said to be *totally umbilical* (resp. *geodesic*). If there exists a scalar field $\tau$ on $M$ such that $A_HX=\tau X$ for any vector field $X$ on $M$, then $M$ is called a *pseudo-umbilical submanifold*. A normal vector field $N$ or the endomorphism $f^{\perp}$ of the normal bundle $T^{\perp}(M)$ is said to be *parallel in the normal bundle* if $\nabla_X^{\perp}N=0$ or $\nabla_X^{\perp}f^{\perp}=0$ for any vector field $X$ on $M$.

Now we define the subbundle $D^{\perp}$ of the normal bundle $T^{\perp}(M)$ by

$$D^{\perp}=\text{span}\{N_1, N_2, \ldots, N_r\}$$

and denote by $D_c^\perp$ the orthogonal complement of $D^\perp$. If each normal vector field $N_a$ in $D^\perp$ is an umbilical section on $M$ and all of them are not geodesic sections on $M$, then the subbundle $D^\perp$ is called an ***umbilical r-plane section*** on $M$. If each $N_a$ is parallel in the normal bundle, then $D^\perp$ is called a ***parallel r-plane section*** on $M$, If $D^\perp$ is an umbilical as well as a parallel $r$-plane section on $M$, then we call it a ***concurrent r-plane section*** on $M$. The following lemma is easily seen and justifies the preceding terminologies of $r$-plane sections.

LEMMA 2.1. *Let $N$ be any vector field in the subbundle $D^\perp$.*

(1) *If all the orthonormal vector fields in $D^\perp$ are umbilical sections on $M$, then so is $N$.*

(2) *If all the orthonormal vector fields in $D^\perp$ are parallel in the normal bundle, then $\nabla_X{}^\perp N$ belongs to $D^\perp$.*

If the subbundle $D^\perp$ of the normal bundle is an umbilical (resp. a parallel or concurrent) $r$-plane section on $M$, then $M$ is said to be a *submanifold with an umbilical* (resp. a *parallel* or a *concurrent*) *r-plane section.*

## 3. Mean curvature vector field

Let $M$ be a submanifold immersed in a Kaehlerian manifold $\tilde{M}$ such that the rank of the map $v : T^\perp(M) \longrightarrow T(M)$ is equal to a constant $r$ almost everywhere on $M$.

If we substitute (2.6) into (2.5), then we obtain

$$f^\perp h(X, Y) - h(fX, Y) = (g(fA_aY, X) - \lambda_{ab}g(A_bX, Y) - L_{ab}(Y)v_b(X))N_a + v_a(X)\nabla_Y{}^\perp N_a$$

for any vector fields $X$ and $Y$ on $M$. Let $E_1, E_2, \ldots, E_n$ be an orthonormal basis for $M$. Then, by taking account of

$$h(X, Y) = h_p(X, Y)C_p = g(A_pX, Y)C_p,$$

the skew-symmetrization of $f$ and symmetrization of $A$, we get

$$g(A_pE_i, E_i)f^\perp C_p = -(\lambda_{ab}g(A_bE_i, E_i) + L_{ab}(E_i)v_b(E_i))N_a + v_a(E_i)\nabla_{E_i}{}^\perp N_a$$

or, summing over $i$,

$$f^\perp H = -\lambda_{ab}\tau_b N_a + H^\perp, \tag{3.1}$$

where we have put

$$H^\perp = (1/n)(\nabla_{V_a}{}^\perp N_a - L_{ab}(V_b)N_a). \tag{3.2}$$

Applying $f^\perp$ to (3.1) and using of (1.12) and (1.15), we have

(3.3) $$H=\tau_a N_a - f^\perp H^\perp.$$

It follows from (3.2) that $H^\perp$ is a vector field in $D_c^\perp$. Thus we have

PROPOSITION 3.1. *Let $M$ be a submanifold in Kaehlerian manifolds such that the rank of the map $v: T^\perp(M) \longrightarrow T(M)$ is equal to a constant $r$ almost everywhere on $M$. Then the mean curvature vector field $H$ of $M$ is given by*

$$H=\tau_a N_a - f^\perp H^\perp,$$

*where $H^\perp$ is a vector field in the subbundle $D_c^\perp$.*

Now we assume that the endomorphism $f^\perp$ of the normal bundle is parallel in the normal bundle. Then it is immediate from (2.8) that

$$v_a(A_N X)N_a = G(N, N_a)h(X, V_a)$$

for any vector field $X$ on $M$ and normal vector field $N$ to $M$. Putting $N=f^\perp H^\perp$ or $H^\perp$ into this relation, we get

(3.4) $$v_a(A_{f^\perp H^\perp}X)=0 \quad \text{or} \quad v_a(A_{H^\perp}X)=0.$$

On the other hand, the relation (2.7) is reduced to

(3.5) $$fA_{f^\perp H^\perp}X + A_{H^\perp}X = -G(f^\perp \nabla_X{}^\perp H^\perp, N_a)V_a.$$

Since $A_{f^\perp H^\perp}X$ and $A_{H^\perp}X$ are tangent vector fields in $D_c$ by (3.4), then so is $fA_{f^\perp H^\perp}X$. Combining this result with (3.5), we find

$$G(f^\perp \nabla_X{}^\perp H^\perp, N_a)=0$$

for any index $a$, which means that $f^\perp \nabla_X{}^\perp H^\perp$ is a normal vector field in the subbundle $D_c^\perp$. Therefore it follows from $f^\perp D_c^\perp \subset D_c^\perp$ and (1.15) that $\nabla_X{}^\perp H^\perp$ belongs to $D_c^\perp$. Moreover the relation (3.5) is reduced to $A_{f^\perp H^\perp}X = fA_{H^\perp}X$ or equivalently

$$g(A_{f^\perp H^\perp}X, Y) = g(fA_{H^\perp}X, Y)$$

for any vector fields $X$ and $Y$ on $M$. For an orthonormal basis $(E_i)$ for $M$, if we put $X=Y=E_i$ and sum over $i$, we get $\theta=0$, where

$$\theta = -(1/n)G(A_{f^\perp H^\perp}E_i, E_i)$$

and it is the mean curvature restricted to the subbundle $D_c^\perp$. Since it is easily seen that $\theta=G(H^\perp, H^\perp)=|H^\perp|^2$, then we can find $H^\perp=0$ identically. Thus we can state

PROPOSITION 3.2. *Let $M$ be a submanifold in Kaehlerian manifolds* *such that the rank of the map* $v : T^{\perp}(M) \longrightarrow T(M)$ *is equal to a constant* $r$ *almost everywhere on* $M$. *If the endomorphism* $f^{\perp}$ *is parallel in the normal bundle, then the mean curvature vector field* $H$ *of* $M$ *is given by*

$$H=\tau_a N_a. \tag{3.6}$$

If the subbundle $D^{\perp}$ of the normal bundle $T^{\perp}(M)$ is a parallel $r$-plane section on $M$, then we have $\nabla_X{}^{\perp}N_a=0$ and $L_{ab}(X)N_b=0$ for any vector field $X$ on $M$ and index $a$. Thus the following is immediate from Theorem 3.1.

COROLLARY 3.3. *Let* $M$ *be a submanifold with a parallel* $r$*-plane section in Kaehlerian manifolds. Then the mean curvature vector field* $H$ *of* $M$ *is given by* (3.6).

## 4. Concurrent $r$-plane section

In this Paragraph, we consider a submanifold $M$ with a concurrent $r$-plane section $D^{\perp}$ in a Kaehlerian manifold $\tilde{M}$. Then, for any vector fields $N_a \in D^{\perp}$ and $X$ on $M$, we have

$$A_aX=\tau_aX, \quad \nabla_X{}^{\perp}N_a=0,$$

where $\tau_a$ is the mean curvature belonging to $N_a$. Therefore the equations (2.4), (2.6) and (2.9) are reduced to

$$(\nabla_Y f)X=\tau_a(v_a(X)Y-g(X,Y)V_a), \tag{4.1}$$

$$\nabla_X V_a=\tau_b\lambda_{ba}X+\tau_a fX, \tag{4.2}$$

$$X\lambda_{ab}=\tau_b v_a(X)-\tau_a v_b(X). \tag{4.3}$$

By Corollary 3.3, the mean curvature vector field $H$ of $M$ is given by (3.6). Therefore it is easily seen that

$$A_HX=|H|^2X, \tag{4.4}$$

where $|H|$ is the mean curvature of $M$. Thus we have

THEOREM 4.1. *Let* $M$ *be a submanifold with a concurrent* $r$*-plane section in Kaehlerian manifolds. Then* $M$ *is a pseudo-umbilical submanifold.*

Now we shall prove the following lemma.

LEMMA 4.2. *The gradient vector field of each mean curvature* $\tau_a$ *belonging to* $N_a$ *is represented by a linear combination of the vector fields in the distribution* $D$ *of* $T(M)$ *only.*

*Proof*. By a straightforward computation, we obtain

$$\nabla_Y \nabla_X V_a = -(\tau_b Y\lambda_{ab} + \lambda_{ab} Y\tau_b) X + (Y\tau_a) fX + \tau_a \tau_b (v_b(X) Y - g(X, Y) V_b) - \lambda_{ab}\tau_b \nabla_Y X + \tau_a f \nabla_Y X$$

from the equations (4.1) and (4.2), which implies that

$$R(X, Y) V_a = (\tau_b Y\lambda_{ab} + \lambda_{ab} Y\tau_b + \tau_a \tau_b v_b(Y)) X - (Y\tau_a) fX - (\tau_b X\lambda_{ab} + \lambda_{ab} X\tau_b + \tau_a \tau_b v_b(X)) Y + (X\tau_a) fY,$$

where $R$ is the curvature tensor of $M$. Using the first Bianchi identity, we get

$$(4.5) \qquad (X\tau_a) g(fY, Z) + (Y\tau_a) g(fZ, X) + (Z\tau_a) g(fX, Y) = 0$$

for any vector fields $X$, $Y$ and $Z$ on $M$. In terms of an orthonormal basis $(E_1)$ for $M$, if we put $Y = E_i$ and $Z = fE_i$ into (4.5) and take account of (1.10) and (1.13), we have

$$(n - r - 2 + \lambda_{bc}\lambda_{bc}) X\tau_a + 2(V_b \tau_a) v_b(X) = 0.$$

Thus the above equation completes the proof.

By virtue of Lemma 4.2, we can consider the case where there exists a pair of mean curvature $\tau_a$ and $\tau_b$ belonging to $N_a$ and $N_b$ in $D^\perp$ such that $d\tau_a = \theta V_b$ and $d\tau_b = \theta V_a$ for a scalar field $\theta$ on $M$. Under the consideration, we shall prove

LEMMA 4.3. *If there is a pair of mean curvatures belonging to $N_a$ and $N_b$ such that their gradient vector fields are the same scalar multiple of $V_b$ and $V_a$ in $D$ respectively, then the equation*

$$(4.6) \qquad \nabla_X d\lambda_{ab} = -(\tau_a \tau_c \lambda_{cb} - \tau_b \tau_c \lambda_{ca}) X$$

*is satisfied for any vector field $X$ on $M$.*

*Proof*. Differentiating (4.3) covariantly along $M$ and using (4.2), we obtain

$$YX\lambda_{ab} = (Y\tau_b) v_a(X) - (Y\tau_a) v_b(X) - (\tau_a \tau_c \lambda_{cb} - \tau_b \tau_c \lambda_{ca}) g(X, Y) + (\nabla_Y X) \lambda_{ab}$$

for any vector fields $X$ and $Y$ on $M$, or equivalently

$$(4.7) \qquad \nabla_X d\lambda_{ab} = (X\tau_b) V_a - (X\tau_a) V_b - (\tau_a \tau_c \lambda_{cb} - \tau_b \tau_c \lambda_{ca}) X.$$

Since there exist $\tau_a$ and $\tau_b$ such that $X\tau_a = \theta V_b$ and $X\tau_b = \theta V_a$ by hypothesis, the equation (4.7) is reduced to (4.6).

On a Riemannian manifold, a scalar field $\lambda$ satisfying

$$(4.8) \qquad \nabla_X d\lambda = \phi X$$

for a scalar field $\phi$ and any vector field $X$ is said to be *concircular*.

The following Theorem $A$ is well-known.

THEOREM A ([13, 14]). *If an $n(\geq 2)$-dimensional complete Riemannian manifold $M$ admits a concircular scalar field satisfying* (4.8), *then $M$ is conformal to one of*

(1) *a warped product $I\times\overline{M}$ of an open interval $I$ of a straight line and an $(n-1)$-dimensional complete Riemannian manifold $\overline{M}$,*

(2) *a Euclidean space,*

(3) *an ordinary sphere.*

The scalar field $\lambda_{ab}$ on our submanifold $M$ is a concircular one. Thus, combining Lemma 4.3 and Theorem $A$, we state

THEOREM 4.4. *Let $M$ be an $n$-dimensional complete submanifold with a concurrent $r(\geq 2)$-plane section $D^{\perp}$ in Kaehlerian manifolds. If there is a pair of mean curvatures belonging to $N_a$ and $N_b$ in $D^{\perp}$ such that their gradient vector fields are the same scalar multiple of $V_b$ and $V_a$ in the distribution $D$ respectively, then $M$ is conformal to one of*

(1) *a warped product $I\times\overline{M}$ of an open interval $I$ of a straight line and an $(n-1)$-dimensional complete Riemannian manifold $\overline{M}$,*

(2) *a Euclidean space,*

(3) *an ordinary sphere.*

Finally we suppose that the mean curvature vector field $H$ of $M$ is parallel in the normal bundle. Then we see that each mean curvature $\tau_a$ belonging to $N_a$ is a constant. Moreover, applying $\tau_b$ to the equation (4.7) and summing over $b$, we have

$$\nabla_X d\tau_b\lambda_{ba}=-|H|^2\tau_b\lambda_{ba}X \tag{4.9}$$

for any vector field $X$ on $M$, which shows that $\tau_b\lambda_{ba}$ is a special concircular scalar field on $M$. As for a special concircular scalar field on a Riemannian manifold, the following Theorem $B$ is well-known and due to Y. Tashiro, M. Obata and S. Tanno.

THEOREM B ([8, 12, 13]). *Let $M$ be an $n(\geq 2)$-dimensional complete, connected and simply connected Riemannian manifold. Then $M$ is isometric to an ordinary sphere if and only if $M$ admits a non-trivial solution $\lambda$ of either the equation*

$$\nabla_X d\lambda=-k\lambda X,$$

*or*

$$(4.10)\qquad \nabla\nabla\omega(X;Y;Z)+k^2(2\omega(Z)g(X,Y)+\omega(Y)g(Z,X)+\omega(X)g(Y,Z))=0$$

*for a positive constant $k$ and any vector fields $X, Y$ and $Z$ on $M$, where $\omega$ is a 1-form on $M$ defined by*

$$(4.11)\qquad \omega=d\lambda.$$

The existence of non-trivial solutions $\tau_b\lambda_{ba}$ of the equation (4.9) is supported by the following Lemma.

LEMMA 4.5. *Suppose that the mean curvature vector field $H$ of $M$ is parallel in the normal bundle. If there are two non-zero mean curvatures $\tau_a$ and $\tau_b$ belonging to $N_a$ and $N_b$ in the $r$-plane section $D^\perp$, then the scalar fields $\tau_c\lambda_{ca}$ and $\tau_c\lambda_{cb}$ are not constants. Moreover, if there is at least one non-zero constant mean curvature belonging to a unit normal vector field in $D^\perp$, there is a non-constant scalar field $\tau_c\lambda_{ca}$.*

*Proof*. First of all, we notice that $\tau_c\lambda_{ca}=G(JH,N_a)$. Assume that $\tau_a$ and $\tau_b$ are non-zero, but the scalar field $\tau_c\lambda_{ca}$ is a constant. Then we have

$$G(J\tilde{\nabla}_XH,N_a)+G(JH,\tilde{\nabla}_XN_a)=-G(JA_HX,N_a)-G(JH,A_aX)=0,$$

which implies, by Theorem 3.1, that

$$|H|^2V_a+\tau_a\tau_cV_c=0.$$

Since $V_a$'s are linearly independent, we obtain

$$|H|^2=\sum\tau_a{}^2 \text{ and } \tau_a\tau_b=\tau_a\tau_c=0$$

for $c\neq a,b$, which shows that $\tau_b=0$. This contradicts to the assumption and hence the scalar field $\tau_c\lambda_{ca}$ is not constant.

To prove the remaining part of the lemma, it suffices to consider the case where $\tau_1$ is non-zero constant and $\tau_a=0$ for $a\neq 1$. Since the relation $X\lambda_{1b}=\tau_bv_1(X)-\tau_1v_b(X)$ is satisfied by (4.3), it is easily seen that

$$X\tau_1\lambda_{1b}=-|H|^2v_b(X),$$

which implies that the scalar fields $\tau_1\lambda_{1b}$ is not a constant.

Combining Theorem B with Lemma 4.5, we can state

THEOREM 4.6. *Let $M$ be a complete, connected and simply connected submanifold with a concurrent $r(\geq 1)$-plane section $D^\perp$ in Kaehlerian manifolds. If the mean curvature vector field $H$ of $M$ is parallel in the $r$-plane section $D^\perp$, then $M$ is isometric to an ordinary sphere.*

By an *extrinsic sphere* we mean a totally umbilical submanifold with non-zero parallel mean curvature vector field (see [3] or [16]). The following is immediate from Theorem 4.6 and generalization of a theorem due to B. Y. Chen ([3]).

COROLLARY 4.7. *A complete, connected and simply connected extrinsic sphere with a parallel r-plane section* $D^{\perp}$ *in Kaehlerian manifolds is isometric to an ordinary sphere.*

## 5. Umbilical *r*-plane section

In this Paragraph, we shall consider a submanifold $M$ with an umbilical $r$-plane section $D^{\perp}$ in Kaehlerian manifolds. Then we have $A_aX=\tau_aX$ for any vector fields $N_a\in D^{\perp}$ and $X$ on $M$. The equations (2.4), (2.6), (2.8) and (2.9) are reduced to

(5.1) $\quad (\nabla_Y f)X=\tau_a v_a(X)Y-\tau_a g(X, Y)V_a,$

(5.2) $\quad \nabla_X V_a=\tau_b\lambda_{ba}X+\tau_a fX-L_{ab}(X)V_b,$

(5.3) $\quad (\nabla_X{}^{\perp}f^{\perp})N_a=h(X, V_a)-\tau_a v_b(X)V_b,$

(5.4) $\quad X\lambda_{ab}=\tau_b v_a(X)-\tau_a v_b(X)-\lambda_{ac}L_{bc}(X)+\lambda_{bc}L_{ac}(X)$

for any vector fields $X$ and $Y$ on $M$. If we define a vector field $V$ on $M$ by

(5.5) $$V=\tau_a V_a$$

and denote the associated 1-form of $V$ by $v$, then it follows from (5.2) that

(5.6) $$\nabla_X V=(X\tau_a-\tau_b L_{ba}(X))V_a+\tau_a\tau_a fX.$$

We shall prove the following lemmas.

LEMMA 5.1. *If the mean curvature vector field $H$ of $M$ is parallel in the normal bundle and belongs to the r-plane section $D^{\perp}$, then each mean curvature $\tau_a$ belonging to $N_a$ is a constant and each scalar field $\tau_b L_{ba}(X)$ vanishes identically on $M$. Moreover the vector field $V$ defined by* (5.5) *is a Killing one.*

*Proof.* Since $H\in D^{\perp}$ *and* $\nabla_X{}^{\perp}H=0$, then it is easily seen from Theorem 3.1 that $H=\tau_a N_a$ and $\tau_a$ is a constant for each $a$. Differentiating this relation covariantly, we also find $\tau_b L_{ba}(X)=0$ for each $a$. Therefore the equation (5.6) is reduced to

(5.7) $$\nabla_X V=|H|^2 fX,$$

which implies that $V$ is a Killing vector field on $M$.

LEMMA 5.2. *Under the assumptions of Lemma* 5.1, *the Killing vector field* $V$ *on* $M$ *satisfies the equation*

$$|H|^2(\nabla_Y\nabla_X V-\nabla_{\nabla_Y X}V)=v(X)Y-g(X,Y)V \tag{5.8}$$

*or*

$$\nabla\nabla\omega(X;Y;Z)+|H|^2(2\omega(Z)g(X,Y) +\omega(Y)g(Z,X)+\omega(X)g(Y,Z))=0 \tag{5.9}$$

*for any vector fields* $X, Y$ *and* $Z$ *on* $M$, *where* $\omega$ *is a* 1*-form given by*

$$\omega=d|V|^2. \tag{5.10}$$

*Proof*. Differentiating (5.7) covariantly along $M$ and taking account of (5.1), we obtain

$$\nabla_Y\nabla_X V=|H|^2(v(X)Y-g(X,Y)+f\nabla_Y X).$$

Therefore the equation (5.8) follows from this equation and (5.7).

If the length $|V|$ of the Killing vector field $V$ is not a constant, then the 1-form $\omega$ given by (5.10) is well-defined. It follows from the 1-form $\omega$ and the equation (5.7) that

$$\omega(X)=2g(\nabla_X V, V)=2|H|^2v(fX). \tag{5.11}$$

Using the equations (5.1), (5.7) and (5.11), we have

$$(\nabla_Y\omega)(X)=2|H|^2(v(X)v(Y)-|V|^2g(X,Y)+|H|^2g(fX,fY))$$

for any vector fields $X$ and $Y$ on $M$. By a simple computation, we can verify the equation (5.9).

The following Theorem $C$ is well-known and due to M. Okumura.

THEOREM C ([9]). *If a Riemannian manifold* $M$ *admits a Killing vector field* $V$ *of constant length satisfying the equation* (5.8), *then* $M$ *is homothetic to a Sasakian manifold.*

We now suppose that the submanifold $M$ is complete, connected and simply connected. If the length $|V|$ of the Killing vector field $V$ is non-trivial, that is, $|V|$ is not a constant, then $M$ is isometric to an ordinary sphere by virtue of Lemma 5.2 and Theorem $B$ stated in Paragraph 4. If the length $|V|$ is a constant, then Theorem $C$ together with Lemma 5.2 show that $M$ is homothetic to a Sasakian manifold (as to a Sasakian manifold, see [9] or [10]). Summing up these results and Lemma 5.1, we can state

THEOREM 5.3. *Let $M$ be a complete, connected and simply connected submanifold with an umbilical r-plane section in Kaehlerian manifolds. If the mean curvature vector field of $M$ is parallel in the normal bundle and belongs to the r-plane section, then $M$ is one of the followings:*

(1) *$M$ is isometric to an ordinary sphere;*

(2) *$M$ is homothetic to a Sasakian manifold.*

Finally we suppose that both the mean curvature vector field $H$ of $M$ and the endomorphism $f^{\perp} : T^{\perp}(M) \longrightarrow T^{\perp}(M)$ are parallel in the normal bundle. Then it is easily verified from Theorem 3.2 that the assumptions of Lemma 5.1 are satisfied. Therefore, if the length $|V|$ of the Killing vector field $V$ is not a constant, $M$ is isometric to an ordinary sphere by Theorem 5.1. In the case where $|V|$ is a constant, it follows from (5.2) and (5.7) that

$$v(fX)=0 \tag{5.12}$$

or, from (1.11) and (5.12),

$$fV=0 \tag{5.13}$$

and $\tau_b\lambda_{ba}=0$ for all $a$.

If we apply $\tau_a$ to the equation (5.4) and sum over $a$, then we find

$$\tau_b v(X)=|H|^2 v_b(X),$$

which implies that all the mean curvatures belonging to $N_a$'s except one vanish identically, say $\tau_1\neq 0$. Therefore we have $V=\tau_1 V_1$ and $|H|=\tau_1$.

Since $f^{\perp}$ is parallel in the normal bundle, it follows from (5.3) that

$$h(X, V_1)=\tau_1 v_b(X) N_b$$

or, applying $N_a$ to this relation,

$$\tau_a v_1(X)=\tau_1 v_a(X),$$

which implies that $V_a$ $(a\neq 1)$ must be vanished, that is, $M$ must be a $CR$-submanifold. We may assume that the constant $\tau_1$ is equal to 1. Hence $V$ is a unit vector field on $M$, that is,

$$v(V)=1. \tag{5.14}$$

The relations (1.10) and (1.13) are reduced to

$$f^2X=-X+v(X)V, \tag{5.15}$$

$$g(fX,fY)=g(X,Y)-v(X)v(Y) \tag{5.16}$$

for any vector fields $X$ and $Y$ on $M$. It follows from (5.7) that

$$\nabla_X V=fX, \tag{5.17}$$

and from (5.1) that

(5.18) $$(\nabla_Y f)X = v(X)Y - g(X, Y)V.$$

The equations (5.12) to (5.18) show that $M$ is just a Sasakian manifold. Thus we can state

THEOREM 5.4. *Let $M$ be a complete, connected and simply connected submanifold with an umbilical r-plane section immersed in Kaehlerian manifolds. If both the mean curvature vector field of $M$ and the endomorphism $f^{\perp}$ are parallel in the normal bundle, then $M$ is one of the followings:*

(1) *$M$ is isometric to an ordinary sphere;*

(2) *$M$ is a Sasakian manifold.*

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