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ON FINITE SIMPLE GROUPS —THE UNITARY GROUPS $U_4(q)$

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1. Introduction

Any finite simple group of Lie type has a (B, N)-pair, and such a simple group can be studied by using its (B, N)-pair structure. For example, the projective special unitary group $U_n(q)$, where q is a prime power, is isomorphic to the twisted Chevalley group ${}^2A_{n-1}(q)$ of type A_{n-1}, and so this group can be studied in this manner.

In this paper we study the simple group $U_4(q)$ by using its (B, N)-pair structure. As a matter of fact, this study will be continued to the study on $U_n(q)$. We will explicitly determine elements of ${}^2A_3(q)$ and give a specific isomorphism of $U_4(q)$ onto ${}^2A_3(q)$. Using this information, we will study some special subgroups contained in maximal parabolic subgroups of $U_4(q)$ and find all elements of order p, where $q=p^e$ and p is a prime. We will also explicitly determine the structure of the centralizers of involutions in $U_4(q)$. Note that

$$|U_4(q)| = \frac{1}{d} q^6 (q^2-1)(q^3+1)(q^4-1),$$

where $d=(4,\ q+1)$, and that

$$d=1 \iff q=2^e \text{ and } p=2;$$
$$d=2 \iff q\equiv 1 \pmod 4; \quad d=4 \iff q\equiv -1 \pmod 4.$$

The determination of involutions and their centralizers in Chevalley groups over a finite field of characteristic 2 has been studied in [1]. And a characterization of $U_4(q)$, where q is odd, has been done in [4] by using matrix presentation. In this paper we treat $U_4(q)$ as the twisted Chevalley group ${}^2A_3(q)$ and every element of subgroups are explicitly expressed. This paper contains some results in the author's paper [3].

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This paper is organized as follows. In section 2 detailed description for the (B, N)-pair structure of ${}^2A_3(q)$ is given. An explicit isomorphism of $U_4(q)$ onto ${}^2A_3(q)$ is also given. In section 3 we prove properties of some special subgroups which are contained in maximal parabolic subgroups of $U_4(q)$. And elements of order p in $U_4(q)$ is also determined. In section 4 and section 5 the structure of the centralizers of involutions in $U_4(q)$ are explicitly determined, where $q\equiv 1 \pmod 4$ or $q\equiv -1 \pmod 4$.

The notation and terminology in this paper are standard. They are taken from [6] for the general finite groups and from [2] for the Chevalley groups.

2. The groups $U_4(q)$

Let F be a finite field with q^2 elements, where $q=p^e$ and p is a prime. For each element $\alpha\in F$ define $\bar{\alpha}$ by $\bar{\alpha}=\alpha^q$. Then the map $\sigma : F \longrightarrow F$, $\sigma(\alpha)=\bar{\alpha}$, is an automorphism of order 2, and $F_0=\{\alpha \in F \mid \alpha=\bar{\alpha}\}$ is a subfield of F with q elements.

Let f be a non-degenerate Hermitian product on the four-dimensional vector space F^4 defined by

$$f((\alpha_1, \alpha_2, \alpha_3, \alpha_4),\ (\beta_1, \beta_2, \beta_3, \beta_4))=\alpha_1\beta_4-\alpha_2\beta_3+\alpha_3\beta_2-\alpha_4\beta_1,$$

and let $J \in Mat_4(F)$ be the matrix associated with f. Then the general unitary group $GU_4(q)$ may be identified with the set of all matrices $(\alpha_{ij}) \in Mat_4(F)$ such that $(\alpha_{ij})^*J(\alpha_{ij})=J$, where $(\alpha_{ij})^*=(\bar{\alpha}_{ji})$. Thus

$$SU_4(q)=\{A \in GU_4(q) \mid \det A=1\}, \quad U_4(q)=PSU_4(q)=SU_4(q)/Z,$$

where $Z=\{\mathrm{diag}\{\lambda, \lambda, \lambda, \lambda\} \mid \lambda\bar{\lambda}=1, \lambda^4=1\}$, and Z is cyclic of order $d=(4,\ q+1)$.

The group $U_4(q)$ is isomorphic to the twisted Chevalley group ${}^2A_3(q)$ of type A_3. We will use the same notation as in [2] to define elements of ${}^2A_3(q)$. Let Φ be the set of roots of the simple Lie algebra $\mathcal{L}$ of type A_3 over the complex number field, and let $\prod=\{a, b, c\}$ be a fundamental system in Φ. The positive system of roots in Φ is then $\Phi^+=\{a, b, c,\ a+b,\ b+c,\ a+b+c\}$. Let $\mathcal{B}=\{h_r,\ r \in \prod;\ e_r,\ r \in \Phi\}$ be a standard Chevalley basis for $\mathcal{L}$. For $r, s \in \Phi$ we define $N_{r,s}$ by $[e_r, e_s]=N_{r,s}e_{r+s}$. Then $N_{r,s}$ is either ± 1 or 0 according as $r+s$ is a root or not.

Let $x_r(\alpha)$, $h_r(\lambda)$, $h(\chi)$ and n_r be elements of the Chevalley group

$A_3(q^2)$ as defined in [2], where $r \in \Phi^+$ and α, $\lambda \in F$. The elements $h_r(\lambda)$ and n_r act on $\mathscr{B}$ as follows:

$$(*) \qquad \begin{array}{lll} h_r(\lambda)\cdot h_s = h_s, & n_r \cdot h_s = h_{w_r}(s); & s \in \prod, \\ h_r(\lambda)\cdot e_s = \lambda^{A_{rs}} e_s & n_r \cdot e_s = \eta_{r,s} e_{w_r(s)}, & s \in \Phi, \end{array}$$

where w_r is the reflection determined by r and $A_{rs} = \dfrac{2(r,s)}{(r,r)}$. The signs of the structure constants $N_{r,s}$ may be chosen arbitrarily for the pairs (a, b), (b, c), $(a, b+c)$, and then the structure constants for all pairs are uniquely determined. We will set

$$(**) \qquad N_{a,b}=1, \quad N_{b,c}=-1, \quad N_{a,b+c}=-1.$$

Then it is easy to prove the following three propositions.

(2.1) *We have* $N_{c,b}=N_{b+c,a}=N_{a+b,c}=1$, $N_{b,a}=N_{c,a+b}=-1$. *If* $r, s, r+s \in \Phi^+$, *then* $\eta_{r,s}=N_{r,s}$ *and* $\eta_{r,r}=\eta_{r,-r}=-1$.

We have

$$\eta_{a,c}=\eta_{a,a+b+c}=\eta_{b,a+b}=\eta_{b,b+c}=\eta_{b,a+b+c}=1,$$
$$\eta_{c,a}=\eta_{c,a+b+c}=1, \quad \eta_{a,a+b}=\eta_{c,b+c}=-1.$$

(2.2) *For any* $r \in \Phi^+$ *we have*

$$x_r(\alpha)x_r(\beta)=x_r(\alpha+\beta), \quad h_r(\lambda)h_r(\mu)=h_r(\lambda\mu).$$

If $r, s \in \Phi^+$ *are distinct, then*

$$[x_r(\alpha), x_s(\beta)]=x_{r+s}(N_{r,s}\alpha\beta), \quad [h_r(\lambda), h_s(\mu)]=1.$$

And for any $r, s \in \Phi^+$ *we have*

$$h_r(\lambda)x_s(\alpha)h_r(\lambda)^{-1}=x_s(\lambda^{A_{rs}}\alpha).$$

(2.3) *For any* $r, s \in \Phi^+$ *we have*

$$n_r x_s(\alpha)n_r^{-1}=x_{w_r(s)}(\eta_{r,s}\alpha), \quad n_r h_s(\lambda)n_r^{-1}=h(\chi),$$

where χ *is a character defined by* $\chi(t)=\lambda^T$ *with* $T=A_{s,w_r(t)}$.

The nontrivial symmetry ρ of the Dynkin diagram for A_3 is given by $\rho(a)=c$, $\rho(b)=b$, $\rho(c)=a$. Thus $S_1=\{a, c\}$ and $S_2=\{b\}$ are orbits in $\prod$ under the action of ρ. Define $w_1=w_a w_c$ and $w_2=w_b$. Then

$$\begin{array}{lll} w_1(a)=-a, & w_1(b)=a+b+c, & w_1(c)=-c, \\ w_2(a)=a+b, & w_2(b)=-b, & w_2(c)=b+c. \end{array}$$

Therefore, w_1 and w_2 are involutions satisfying $(w_1w_2)^4=1$. Set $S_3=\{a+b, b+c\}$ and $S_4=\{a+b+c\}$. Then we have

$$w_1(S_1)=-S_1, \quad w_1(S_2)=S_4, \quad w_1(S_3)=S_3,$$

$$w_2(S_1)=S_3, \qquad w_2(S_2)=-S_2, \quad w_2(S_4)=S_4.$$

Hence the Weyl group W of the twisted Chevalley group ${}^2A_3(q)$ is generated by w_1 and w_2, and $W=\langle w_1, w_2\rangle$ is dihedral of order 8.

Now we define elements of ${}^2A_3(q)$ as follows:

$$x_1(\alpha)=x_a(\alpha)x_c(\bar{\alpha}),\ \alpha\in F; \qquad x_2(\beta)=x_b(\beta),\ \beta\in F_0;$$
$$x_3(\gamma)=x_{a+b}(\gamma)x_{b+c}(\bar{\gamma}),\ \gamma\in F; \qquad x_4(\delta)=x_{a+b+c}(\delta),\ \delta\in F_0;$$
$$h_1(\mu)=h_a(\mu)h_c(\bar{\mu}),\ \mu\in F^*; \qquad h_2(\lambda)=h_b(\lambda),\ \lambda\in F_0{}^*;$$
$$n_1=n_an_c, \quad n_2=n_b,$$

where $F^*=F-\{0\}$ and $F_0{}^*=F_0-\{0\}$.

Using (*), (**) and (2.1)~(2.3), we can easily prove the following properties of ${}^2A_3(q)$.

(2.4) *Let*

$$U_1=\{x_1(\alpha)\mid\alpha\in F\},\quad U_2=\{x_2(\beta)\mid\beta\in F_0\},$$
$$U_3=\{x_3(\gamma)\mid\gamma\in F\},\quad U_4=\{x_4(\delta)\mid\delta\in F_0\},\quad U=U_1U_2U_3U_4.$$

Then U_1, U_2, U_3, U_4 are elementary abelian p-subgroups such that $|U_1|=|U_3|=q^2$ and $|U_2|=|U_4|=q$. The multiplication in each U_i is given by $x_i(\alpha)x_i(\beta)=x_i(\alpha\beta)$.

The subgroup U is a Sylow p-subgroup of ${}^2A_3(q)$ of order q^6 with center $Z(U)=U_4$. Every element of U is uniquely expressed as a product $x_1(\alpha)x_2(\beta)x_3(\gamma)x_4(\delta)$. We have

$$[x_1(\alpha), x_2(\beta)]=x_3(\alpha\beta)x_4(\alpha\bar{\alpha}\beta),\quad [x_3(\gamma), x_1(\alpha)]=x_4(\alpha\bar{\gamma}+\bar{\alpha}\gamma),$$

and all other types of commutators between elements of the various U_i are trivial.

(2.5) *Let*

$$H_1=\{h_1(\mu)\mid\mu\in F^*\},\quad H_2=\{h_2(\lambda)\mid\lambda\in F_0{}^*\},$$
$$H=H_1H_2, \qquad d=(4, q+1).$$

Then H_1 is a cyclic group isomorphic to $F^*/\langle -1\rangle$ and

$$h_1(\mu)h_1(\mu')=h_1(\mu\mu');\quad h_1(\mu)=h_1(\mu')\Longleftrightarrow\mu=\pm\mu'.$$

The subgroup H_2 is a cyclic group isomorphic to $F_0{}^*$ and

$$h_2(\lambda)h_2(\lambda')=h_2(\lambda\lambda');\qquad h_2(\lambda)=1\Longleftrightarrow\lambda=1.$$

The subgroup H is abelian of order $\frac{1}{d}(q^2-1)(q-1)$, and every element of H can be expressed as a product $h_1(\mu)h_2(\lambda)$.

If $d=1$ or $d=2$, then $H=H_1\times H_2$.

If $d=4$, then $H_1\cap H_2=\{1, h_2(-1)\}$. In general, we have

$$h_1(\mu)h_2(\lambda)=1 \Longleftrightarrow \mu^d=1 \text{ and } \mu^2=\lambda.$$

(2.6) *Let $B=UH$. Then B is the normalizer of U in ${}^2A_3(q)$, and the action of $h=h_1(\mu)h_2(\lambda)\in H$ on U is given by*

$$h^{-1}x_1(\alpha)h=x_1(\mu^{-2}\lambda\alpha), \qquad h^{-1}x_2(\beta)h=x_2(\mu\bar{\mu}\lambda^{-2}\beta),$$
$$h^{-1}x_3(\gamma)h=x_3(\mu^{-1}\bar{\mu}\lambda^{-1}\gamma), \quad h^{-1}x_4(\delta)h=x_4(\mu^{-1}\bar{\mu}^{-1}\delta).$$

(2.7) *Let $M=\langle n_1, n_2\rangle$ and $N=HM$. Then $H=B\cap N$ and H is normal in N. The action of M on H is given by*

$$n_1h_1(\mu)n_1=h_1(\mu^{-1}), \qquad n_1h_2(\lambda)n_1=h_1(\lambda)h_2(\lambda),$$
$$n_2^{-1}h_1(\mu)n_2=h_1(\mu)h_2(\mu\bar{\mu}), \quad n_2^{-1}h_2(\lambda)n_2=h_2(\lambda^{-1}).$$

There is an isomorphism of N/H onto W which sends n_1H into w_1 and n_2H into w_2.

If q is even, then M is a dihedral group of order 8 generated by two involutions n_1 and n_2 satisfying $(n_1n_2)^4=1$. Moreover, $M\cap H=1$. If q is odd, then M is a nonabelian group of order 16 such that

$$n_1^2=1, \quad n_2^2=h_2(-1), \quad (n_1n_2)^4=1,$$
$$Z(M)=\langle h_2(-1)\rangle\times\langle(n_1n_2)^2\rangle, \ M'=\langle[n_1, n_2]\rangle, \ M\cap H=\langle h_2(-1)\rangle.$$

(2.8) *The elements n_1 and n_2 transform the elements of U in the following manner:*

$$n_1x_1(\alpha)n_1=x_1(-\alpha^{-1})h_1(\alpha^{-1})n_1x_1(-\alpha^{-1}), \quad \alpha\neq 0;$$
$$n_2^{-1}x_2(\beta)n_2=x_2(-\beta^{-1})h_2(\beta^{-1})n_2x_2(-\beta^{-1}), \quad \beta\neq 0;$$
$$n_1x_2(\beta)n_1=x_4(-\beta), \ n_1x_3(\gamma)n_1=x_3(-\bar{\gamma}), \ n_1x_4(\delta)n_1=x_2(-\delta),$$
$$n_2^{-1}x_1(\alpha)n_2=x_3(\alpha), \ n_2^{-1}x_3(\gamma)n_2=x_1(-\gamma), \ n_2^{-1}x_4(\delta)n_2=x_4(\delta).$$

(2.9) *The subgroups B and N form a (B, N)-pair of ${}^2A_3(q)$. The group ${}^2A_3(q)$ is the disjoint union of eight double cosets of the form BnB, where n runs through the transversal*

$$\overline{N}=\{1, n_1, n_2, n_1n_2, n_2n_1, n_1n_2n_1, n_2n_1n_1, (n_1n_2)^2\}$$

of $\langle h_2(-1)\rangle$ in N. In fact, we have $BnB=BnU_n$, where U_n is given in the following table:

n :	1	n_1	n_2	n_1n_2	n_2n_1	$n_1n_2n_1$	$n_2n_1n_2$	$(n_1n_2)^2$
U_n :	1	U_1	U_2	U_2U_3	U_1U_4	$U_1U_3U_4$	$U_2U_3U_4$	$U_1U_2U_3U_4$.

Every element of ${}^2A_3(q)$ can be uniquely expressed as a product bny, where $b\in B$, $n\in\overline{N}$ and $y\in U_n$.

(2.10) *There exists an isomorphism of ${}^2A_3(q)$ onto $U_4(q)=SU_4(q)/Z$ which sends*

$$x_1(\alpha) \quad \text{into} \quad \begin{pmatrix} 1 & \alpha & & \\ & 1 & & \\ & & 1 & \bar{\alpha} \\ & & & 1 \end{pmatrix} Z, \qquad x_2(\beta)x_3(\gamma)x_4(\delta) \quad \text{into} \quad \begin{pmatrix} 1 & & \gamma & \delta \\ & 1 & \beta & -\bar{\gamma} \\ & & 1 & \\ & & & 1 \end{pmatrix} Z,$$

$$h_1(\mu) \quad \text{into} \quad \begin{pmatrix} \mu & & & \\ & \mu^{-1} & & \\ & & \bar{\mu} & \\ & & & \bar{\mu}^{-1} \end{pmatrix} Z, \quad h_2(\lambda) \quad \text{into} \quad \begin{pmatrix} 1 & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & 1 \end{pmatrix} Z,$$

$$n_1 \quad \text{into} \quad \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} Z, \quad n_2 \quad \text{into} \quad \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} Z.$$

Note that if q is even then (2.4)~(2.9) are identical with (2.1)~(2.6) in [3].

Using (2.4)~(2.10) we can construct a group which is isomorphic to $SU_4(q)$ in the following way:

(2.11) *Let* $\bar{H}_1 = \{\bar{h}_1(\mu) \mid \mu \in F^*\}$ *be a cyclic group isomorphic to* F^*, *whose multiplication is defined by* $\bar{h}_1(\mu)\bar{h}_1(\mu') = \bar{h}_1(\mu\mu')$. *Let* $\bar{H} = \bar{H}_1 \times H_2$ *and define the relations between* $\bar{h}_1(\mu)$ *and* $x_i(\alpha)$, n_1, n_2 *to be the same as those relations between* $h_1(\mu)$ *and* $x_i(\alpha)$, n_1, n_2 *in* (2.4)~(2.8).

Then the group $\bar{G} = \langle U, \bar{H}, n_1, n_2 \rangle$ *is isomorphic to* $SU_4(q)$. *Moreover, an isomorphism of* $\bar{G}$ *onto* $SU_4(q)$ *can be defined as in* (2.10) *by replacing* $h_1(\mu)$ *by* $\bar{h}_1(\mu)$ *and omitting* Z.

3. Parabolic subgroups and elements of order p

From now on we identify $U_4(q)$ with the twisted Chevalley group ${}^2A_3(q)$, where $q = p^e$ and p is a prime. The letter which is introduced in section 2 will keep its meaning throughout this paper.

In this section we will discuss some properties of maximal parabolic subgroups of $U_4(q)$. We also explicitly determine all the elements of $U_4(q)$ of order p and the centralizers of some elements of order p.

(3.1) *Let*

$$B_1 = B \cup Bn_1U_1, \quad B_2 = B \cup Bn_2U_2.$$

Then B_1 *and* B_2 *are maximal parabolic subgroups of* $U_4(q)$, *and every parabolic subgroups of* $U_4(q)$ *is conjugate to* B, B_1, B_2 *or* $U_4(q)$.

Proof. This follows from the (B, N)-pair structure of $U_4(q)$.

(3.2) Let

$$B_1=B\cup Bn_1U_1,\quad P_1=UH_1\cup UH_1n_1U,\quad P=U_1H_1\cup U_1H_1n_1U_1.$$

Then the following hold:

(1) B_1 *has a maximal normal* p*-subgroup* $U_2U_3U_4$, *which is an elementary abelian* p*-subgroup of order* q^4. *The subgroup* $U_1H\cup U_1Hn_1U_1$ *is a complement of* $U_2U_3U_4$ *in* B_1.

(2) P_1 *is a normal subgroup of* B_1 *such that* $B_1=P_1H_2$ *and* B_1/P_1 *is cyclic. And* P *is a complement of a normal subgroup* $U_2U_3U_4$ *in* P_1.

Moreover, $P\cong PSL_2(q^2)$.

Proof. Using (2.4)$\sim$(2.9) we can easily prove the assertions. Note that

$$h_2(\lambda)^{-1}n_1h_2(\lambda)=h_1(\lambda)n_1.$$

It is easy to show that there exists an isomorphism of the group P onto $PSL_2(q^2)$ which sends $x_1(\alpha), h_1(\mu)$, n_1 into

$$\begin{pmatrix}1 & \alpha\\ & 1\end{pmatrix}Z_1,\quad \begin{pmatrix}\mu & \\ & \mu^{-1}\end{pmatrix}Z_1,\quad \begin{pmatrix} & 1\\ -1 & \end{pmatrix}Z_1,$$

respectively. Here Z_1 denotes the center of $SL_2(q^2)$. Notice that the above matricies are obtained from the corresponding matrices given in (2.10) in a suitable way.

(3.3) *Let*

$$B_2=B\cup Bn_2U_2,\quad P_2=UH_2\cup UH_2n_2U_2,\quad Q=U_2H_2\cup U_2H_2n_2U_2.$$

Then the following hold:

(1) B_2 *has a maximal normal* p*-subgroup* $U_1U_3U_4$ *of order* q^5. *The subgroup* $U_2H\cup U_2Hn_2U_2$ *is a complement of* $U_1U_3U_4$ *in* B_2.

(2) P_2 *is a normal subgroup of* B_2 *such that* $B_2=P_2H_1$ *and* B_2/P_2 *is cyclic. And* Q *is a complement of a normal subgroup* $U_1U_3U_4$ *in* P_2.

(3) P_2 *isomorphic to a subgroup of* $SU_4(q)$ *and*

$$Q\cong SL_2(q)\cong SU_2(q).$$

Proof. The assertions can be proved by using (2.4)$\sim$(2.9). Note that

$$h_1(\mu)^{-1}n_2h_1(\mu)=h_2(\mu\bar{\mu})n_2.$$

By (2.11) we can prove that there exists a monomorphism of P_2 into $SU_4(q)$ which sends $x_1(\alpha)$, $x_2(\beta)x_3(\gamma)x_4(\delta)$, $h_2(\lambda)$, n_2 into

$$\begin{pmatrix}1&\alpha&&\\&1&&\\&&1&\bar\alpha\\&&&1\end{pmatrix},\quad \begin{pmatrix}1&&\gamma&\delta\\&1&\beta&-\bar\gamma\\&&1&\\&&&1\end{pmatrix},\quad \begin{pmatrix}1&&&\\&\lambda&&\\&&\lambda^{-1}&\\&&&1\end{pmatrix},\quad \begin{pmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{pmatrix},$$

respectively. Using this monomorphism we can show that there exists an isomorphism of Q onto $SL_2(q)$ which sends $x_2(\beta)$, $h_2(\lambda)$, n_2 into

$$\begin{pmatrix}1&\beta\\&1\end{pmatrix},\quad \begin{pmatrix}\lambda&\\&\lambda^{-1}\end{pmatrix},\quad \begin{pmatrix}&1\\-1&\end{pmatrix},$$

respectively. Hence we have $Q \cong SL_2(q) \cong SU_2(q)$.

(3.4) *Let*

$$P=U_1H_1 \cup U_1H_1n_1U_1, \quad n_2^{-1}Pn_2=U_3H_3 \cup U_3H_3h_2(-1)n_2n_1n_2U_3,$$

where $H_3=n_2^{-1}H_1n_2=\{h_1(\mu)h_2(\mu\bar\mu) \mid \mu \in F^*\}$. *Then*

$$P \cong n_2^{-1}Pn_2 \cong PSL_2(q^2), \quad \langle P, n_2^{-1}Pn_2\rangle = U_4(q).$$

Proof. The first assertion follows from (3.2). Recall that $n_2^{-1}=h_2(-1)n_2$. Since $[U_3, U_1]=U_4$, $n_1U_4=U_2n_1$, $H_1H_3=H$, the subgroup $\langle P, n_2^{-1}Pn_2\rangle$ contains a maximal parabolic subgroup $B_1=B\cup Bn_1U_1$. Hence we have $\langle P, n_2^{-1}Pn_2\rangle=U_4(q)$, by (3.1).

Compared to the subgroup P, the subgroup $Q=U_2H_2\cup U_2H_2n_2U_2$ has a nice property as we can see in the next proposition. Note that if q is even then $h_2(-1)=h_2(1)=1$, and if q is odd then $h_2(-1)$ is an involution. In any case n_1 is an involution.

(3.5) *Let*

$$Q=U_2H_2\cup U_2H_2n_2U_2, \quad R=n_1Qn_1=U_4H_4\cup U_4H_4n_1n_2n_1U_4,$$

where $H_4=n_1H_2n_1=\{h_1(\lambda)h_2(\lambda) \mid \lambda\in F_0^*\}$. *Then the following hold:*

(1) *We have*

$$Q\cong R\cong SL_2(q)\cong SU_2(q), \quad [Q,R]=1,$$
$$Z(Q)=Z(R)=Q\cap R=H_2\cap H_4=\langle h_2(-1)\rangle.$$

(2) *Let* $E=QR$ *and* $K=H_2H_4$. *Then* E *and* K *are subgroups and we have*

$$E=U_2U_4K\cup U_2U_4Kn_2U_2\cup U_2U_4Kn_1n_2n_1U_4\cup U_2U_4K(n_1n_2)^2U_2U_4.$$

If q *is even, then* $E=Q\times R$ *and* $|E|=q^2(q^2-1)^2$.

If q *is odd, then* E *is the central product of* Q *and* R *such that* $Z(E)=Z(Q)=Z(R)=\langle h_2(-1)\rangle$, *and* $|E|=\frac{1}{2}q^2(p^2-1)^2$.

Proof. It is easy to see that Q and R centralize each other. Now the assertion can be easily proved by (3.3) and (2.4)~(2.8).

We will explicitly determine all elements of order p in $U_4(q)$. Since U is a Sylow p-subgroup of $U_4(q)$, it suffices to find all elements of order p in U.

(3.6) *The following is the list of all elements of order* p *which are contained in the Sylow* p*-subgroup* $U=U_1U_2U_3U_4$.

(i) *the case when* $p=2$.
every nonidentity element in $U_2U_3U_4$, *and*
every element of the form $x_1(\alpha)x_3(\alpha\beta)x_4(\delta)$, *where* $\alpha \in F^*$; $\beta, \delta \in F_0$.

(ii) *the case when* $p=3$.
every nonidentity element in $U_2U_3U_4$, *and*
every nonidentity element in $U_1U_3U_4$.

(iii) *the case when* $p\geq 5$.
every nonidentity element in U.

Proof. By the assertion (3) in (3.3) there exists a monomorphism of U into $SU_4(q)$. For each element a of U, let $A\in SU_4(q)$ be the matrix corresponding to a under this monomorphism. Then it is clear that $(A-I)^4=0$ holds, where I and 0 are the identity matrix and zero matrix, respectively. On the other hand, F is a field of characteristic p. Therefore, if $p\geq 5$ then we have $0=(A-I)^p=A^p-I$, which implies that $a^p=1$. Hence (iii) holds.

An easy calculation yields (i) and (ii).

It is easy to show that $x_4(1)$ and $x_3(1)$ are not conjugate in $U_4(q)$. Hence there at least two conjugacy classes of elements of order p. In particular, if q is even, that is, if $p=2$, then there are exactly two conjugacy classes of involutions in $U_4(q)$. By (3.2) and (3.3) it is easy to prove the following two propositions (*cf.* [3]).

(3.7) *Let* C *be the centralizer of* $x_4(1)$ *in* $U_4(q)$. *Then*

$$C=UL\cup ULn_2U_2\subset B_2=B\cup Bn_2U_2,$$

where $L=\{h_1(\mu)\mid \mu\in F^*, \mu\bar{\mu}=1\}$, *and* C *is of order* $q^6(q+1)^2$ *or* $\frac{1}{2}q^6(q+1)^2$ *according as* q *is even or odd.*

Moreover, $U_1U_3U_4$ *is a maximal normal* p*-subgroup of* C, *and* $U_2L\cup U_2Ln_2U_2$ *is a complement of* $U_1U_3U_4$ *in* C.

(3.8) *Let* C_1 *be the centralizer of* $x_3(1)$ *in* $U_4(q)$. *Then*

$$C_1=\bar{U}_1U_2U_3U_4J\cup\bar{U}_1U_2U_3U_4Jh_2(-1)n_1\bar{U}_1\subset B_1=B\cup Bn_1U_1,$$

where

$$\bar{U}_1=\{x_1(\alpha)\mid\alpha\in F,\ \bar{\alpha}=-\alpha\},$$
$$J=\{h_1(\lambda)\mid\lambda\in F_0^*\}\cup\{h_1(\mu)h_2(-1)\mid\mu\in F^*,\bar{\mu}=-\mu\}.$$

The subgroup $U_2U_3U_4$ *is a maximal normal p-subgroup of* C_1, *and* $\bar{U}_1J\cup\bar{U}_1Jh_2(-1)n_1\bar{U}_1$ *is a complement of* $U_2U_3U_4$ *in* C_1.

If q *is even, then* $|C_1|=q^5(q^2-1)$ *and*

$$\langle\bar{U}_1J\cup\bar{U}_1Jn_1\bar{U}_1\rangle\cong SL_2(q)=PSL_2(q).$$

If $q\equiv1$ *(mod 4), then* $|C_1|=q^5(q^2-1)$.

If $q\equiv-1$ *(mod 4), then* $|C_1|=\frac{1}{2}q^5(q^2-1)$.

4. Involutions and their centralizers in $U_4(q)$, where $q\equiv1$ (mod 4)

In this section we assume that $q\equiv1$ (mod 4), and we will determine the centralizers of involutions in $U_4(q)$. We have

$$d=(4,q+1)=4,\qquad |U_4(q)|=\frac{1}{2}q^6(q^2-1)^2(q^2+1)(q^3+1).$$

(4.1) *Let* C *be the centralizer of the involution* $h_2(-1)$ *in* $U_4(q)$. *Then the following hold*:

(1) *We have*

$$C=U_2U_4H\cup U_2U_4Hn_1\cup U_2U_4Hn_2U_2\cup U_2U_4Hn_1n_2U_2\cup U_2U_4Hn_2n_1U_4$$
$$\cup U_2U_4Hn_1n_2n_1U_4\cup U_2U_4Hn_2n_1n_2U_2U_4\cup U_2U_4H(n_1n_2)^2U_2U_4,$$
$$|C|=q^2(q^2-1)^2(q+1).$$

(2) *Let*

$$D=U_2U_4H\cup U_2U_4Hn_2U_2\cup U_2U_4Hn_1n_2n_1U_4\cup U_2U_4H(n_1n_2)^2U_2U_4,$$
$$E=U_2U_4K\cup U_2U_4Kn_2U_2\cup U_2U_4Kn_1n_2n_1U_4\cup U_2U_4K(n_1n_2)^2U_2U_4,$$
$$Q=U_2H_2\cup U_2H_2n_2U_2,$$
$$R=n_1Qn_1=U_4H_4\cup U_4H_4n_1n_2n_1U_4,\quad H_4=n_1H_2n_1,\quad K=H_2H_4.$$

Then D *and* E *are normal subgroups of* C *such that*

i) $C\supset D\supset E=QR$, $|C:D|=2$, $|D:E|=q+1$,

ii) $C=D\langle n_1\rangle$, $D=EH_1$, *and* D/E *is cyclic of order* $q+1$,

iii) E *is the central product of* Q *and* R, *where*

$$Q\cong R\cong SL_2(q)\cong SU_2(q).$$

Proof. By (2.8) the centralizer of $h_2(-1)$ in U is U_2U_4, and by (2.7), N centralizes $h_2(-1)$. From this fact and (2.9) we can prove the assertion (1). Using (3.5) and (2.5)~(2.9), we can easily prove the assertion (2).

(4.2) *Let C_1 be the centralizer of the involution $h_1(\xi_0)h_2(-1)$ in $U_4(q)$, where ξ_0 is a primitive 4-th root of unity in F^*. Then the following hold*:

(1) *We have*

$$C_1=U_1H\cup U_1Hn_1U_1\cup U_1Hn_2n_1n_2\cup U_1H(n_1n_2)^2U_1,$$
$$|C_1|=q^2(q^4-1)(q-1).$$

(2) *Let*

$$D_1=U_1H\cup U_1Hn_1U_1,\quad P=U_1H_1\cup U_1H_1n_1U_1.$$

Then D_1 and P are normal subgroups of C_1 such that

i) $C_1\supset D_1\supset P$, $|C_1:D_1|=2$, $|D_1:P|=q-1$,

ii) $C_1=D_1\langle n\rangle=P\langle H_2,n\rangle$ *and* $D_1=PH_2$ *are semidirect products, where* $n=(n_1n_2)^2$ *and* $\langle H_2,n\rangle$ *is dihedral of order* $2(q-1)$.

iii) $P\cong PSL_2(q^2)$.

Proof. The centralizer of $h_1(\xi_0)h_2(-1)$ in U is U_1, and the centralizer of this involution in N is $H\langle n_1, n_2n_1n_2\rangle$. Hence the assertion (1) holds. Now we can prove the assertion (2) by using (3.2).

(4.3) *There are exactly two conjugacy classes of involutions in $U_4(q)$ with representatives $h_2(-1)$ and $h_1(\xi_0)h_2(-1)$. Thus the centralizer of any involution in $U_4(q)$ is conjugate in $U_4(q)$ to either C or C_1.*

This result is known (*cf*. [4]). Thus we will sketch the proof. Let 2^m be the highest power of 2 dividing $q-1$, that is, $2^m\|q-1$. Then

$$2^m\||F_0^*|,\quad 2^{m+1}\||F^*|,\quad 2^{2m+3}\||N|,\quad 2^{2m+3}\||U_4(q)|.$$

Hence any Sylow 2-subgroup of N is a Sylow 2-subgroup of $U_4(q)$, and so every involution of $U_4(q)$ is conjugate in $U_4(q)$ to an involution in N. The following is the list of all involutions in N.

(I) $h_2(-1)$;
$h_1(\mu)h_2(-1)n_1,\ \mu\in F^*$;
$h_1(\mu)h_2(\mu\bar{\mu})n_2n_1n_2,\ \mu\in F^*$;
$h_1(\mu)h_2(\lambda)(n_1n_2)^2,\ \lambda\in F_0^*,\ \mu\in F^*,\ \bar{\mu}=-\mu$.

(II) $h_1(\xi_0)$; $h_1(\xi_0)h_2(-1)$;
$h_1(\mu)n_1,\ \mu\in F^*$;
$h_1(\xi_0)h_2(\lambda)n_2,\ \lambda\in F_0^*$;
$h_1(\xi_0\lambda)h_2(\lambda)n_1n_2n_1,\ \lambda\in F_0^*$;
$h_1(\mu)h_2(-\mu\bar{\mu})n_2n_1n_2,\ \mu\in F^*$;
$h_1(\mu)h_2(\lambda)(n_1n_2)^2,\ \mu,\lambda\in F_0^*$.

Now we can show that involutions in (I) are conjugate in $U_4(q)$ to $h_2(-1)$, and involutions in (II) are conjugate to $h_1(\xi_0)h_2(-1)$.

5. Involutions and their centralizers in $U_4(q)$, where $q\equiv-1 \pmod 4$

In this section we assume that $q\equiv-1 \pmod 4$, and we will determine the centralizers of involutions in $U_4(q)$.

Assume that $q\equiv-1 \pmod 4$. Then

$$d=(4,q+1)=4,\quad |U_4(q)|=\frac{1}{4}q^6(q^2-1)^2(q^2+1)(q^3+1).$$

(5.1) *Let C be the centralizer of the involution $h_2(-1)$ in $U_4(q)$. Then the following hold:*

(1) *We have*

$$C=U_2U_4H\cup U_2U_4Hn_1\cup U_2U_4Hn_2U_2\cup U_2U_4Hn_1n_2U_2\cup U_2U_4Hn_2n_1U_4 \cup U_2U_4Hn_1n_2n_1U_4\cup U_2U_4Hn_2n_1n_2U_2U_4\cup U_2U_4H(n_1n_2)^2U_2U_4,$$

$$|C|=\frac{1}{2}q^2(q^2-1)^2(q+1).$$

(2) *Let*

$$D=U_2U_4H\cup U_2U_4Hn_2U_2\cup U_2U_4Hn_1n_2n_1U_4\cup U_2U_4H(n_1n_2)^2U_2U_4,$$
$$E=U_2U_4K\cup U_2U_4Kn_2U_2\cup U_2U_4Kn_1n_2n_1U_4\cup U_2U_4K(n_1n_2)^2U_2U_4,$$
$$Q=U_2H_2\cup U_2H_2n_2U_2,$$
$$R=n_1Qn_1=U_4H_4\cup U_4H_4n_1n_2n_1U_4,$$
$$H_4=n_1H_2n_1,\quad K=H_2H_4.$$

Then D and E are normal subgroups of C such that

i) $C\supset D\supset E=QR,\quad |C:D|=2,\quad |D:E|=\frac{1}{2}(q+1),$

ii) $C=D\langle n_1\rangle=E\langle L,n_1\rangle,\ D=EL,\ E\cap L=\langle h_2(-1)\rangle,$
where $L=\{h_1(\mu)\mid \mu\in F^*,\ \mu^{2(q+1)}=1\}$ is cyclic of order $q+1$, and $\langle L,n_1\rangle$ is a dihedral group of order $2(q+1)$.

iii) *E is the central product of Q and R, where*

$$Q\cong R\cong SL_2(q)\cong SU_2(q).$$

Proof. The proof is essentially same as the proof of (4.1). We will prove the part ii) of (2).

Let ω be a primitive element of F. Then $F^*=\langle\omega\rangle$ and $F_0{}^*=\langle\omega^{q+1}\rangle$. Let $\xi=\omega^{\frac{1}{2}(q-1)}$. Since $q\equiv-1 \pmod 4$, two integers $\frac{1}{2}(q-1)$ and

$q+1$ are relatively prime. Hence we have $F^*=\langle\xi\rangle\times F_0{}^*$. This implies that $H_1=L\times L_1$, where

$$L=\langle h_1(\xi)\rangle=\{h_1(\mu)\mid \mu\in F^*,\ \mu^{2(q+1)}=1\},\ L_1=\{h_1(\lambda)\mid \lambda\in F_0{}^*\}.$$

Therefore, $H=KL$ and $K\cap L=\langle h_2(-1)\rangle$. Note that $h_2(-1)=h_1(\xi_0)$, where ξ_0 is a primitive 4-*th* root of unity which is in $\langle\xi\rangle$. Since the involution n_1 inverts $h_1(\xi)$, the group $\langle L, n_1\rangle$ is a dihedral group of order $2(q+1)$.

Using the above results we can prove the assertion ii).

(5.2) *Assume that* $q\equiv -1 \pmod 8$. *Let* ξ *be a primitive* 8-*th root of unity in* F^* *and let* $\xi_0=\xi^2$. *Then* $h_1(\xi)n_2$ *is an involution of* $U_4(q)$.

Let C_1 *be the centralizer of the involution* $h_1(\xi)n_2$ *in* $U_4(q)$. *Then the following hold*:

(1) *We have*

$$C_1=U_0H_0\cup U_0\mathfrak{I}\cup U_0H_0n_1n_2n_1U_0\cup U_0\mathfrak{I}n_1n_2n_1U_0,$$
$$|C_1|=\frac{1}{4}q^3(q+1)(q^2-1)(q^3+1),$$

where

$$U_0=\{x_1(\alpha)x_3(-\xi_0\alpha)x_4(\delta)\mid \alpha\in F, \delta\in F_0\},$$
$$H_0=\{h_1(\mu)h_2(\lambda)\mid \mu\in F^*, \lambda\in F_0{}^*, \mu\bar{\mu}=\lambda^2\},$$
$$\mathfrak{I}=\{x_2(\beta)h_1(\mu)h_2(\lambda)n_2x_2(\beta)\mid \mu\in F^*, \lambda\in F_0{}^*, \beta\in F_0, \mu\bar{\mu}(\beta^2+1)=\lambda^2\}.$$

(2) *Let*

$$T=H_0\cup\mathfrak{I},$$
$$T_0=\{h_1(\lambda)h_2(\lambda),\ \ x_2(\beta)h_1((\beta-\xi_0)^{-1}\lambda)h_2(\lambda)n_2x_2(\beta)\mid \lambda, \beta\in F_0{}^*\}.$$

Then

$$C_1=U_0T\cup U_0Tn_1n_2n_1U_0,\quad U_0T=U_0H_0\cup U_0\mathfrak{I}.$$

Moreover, U_0T, T, T_0 *are subgroups of a maximal parabolic subgroup* $B_2=B\cup Bn_2U_2$ *such that* $U_0T\supset T\supset T_0$ *and* U_0 *is normal in* U_0T.

(3) *Let*

$$D_1=U_0T_0\cup U_0T_0n_1n_2n_1U_0.$$

Then D_1 *is a normal subgroup of* C_1 *such that*

i) $C_1=D_1J$, $D_1\cap J=\langle h_2(-1)\rangle$, *and*

C_1/D_1 *is cyclic of order* $\frac{1}{4}(q+1)$, *where* $J=\{h_1(\mu)\mid \mu\in F^*,\ \mu\bar{\mu}=1\}$ *is a cyclic group of order* $\frac{1}{2}(q+1)$, *and*

ii) $D_1\cong SU_3(q)$.

Proof. Since $q\equiv-1 \pmod 8$, we have $\xi\bar{\xi}=1$. Hence $h_1(\xi)n_2$ is an involution, by (2.7).

Using (2.9) and (2.10) we can prove the assertion (1). By definition of T, we have $U_0T=U_0H_0\cup U_0\mathfrak{J}$ and it is the centralizer of $h_1(\xi)n_2$ in B_2. Hence U_0T is a subgroup. Now it is easy to show that T and T_0 are subgroups and U_0 is normal in U_0T.

Suppose shat $\mu\bar{\mu}=\lambda^2$, where $\mu\in F^*$ and $\lambda\in F_0{}^*$. If we set $\eta=\mu\lambda^{-1}$, then we have $\mu=\lambda\eta$ and $\eta\bar{\eta}=1$. Suppose that $\mu\bar{\mu}(\beta^2+1)=\lambda^2$ where $\mu\in F^*$, $\lambda\in F_0{}^*$ and $\beta\in F_0$. Set $\eta=\mu\lambda^{-1}(\beta-\xi_0)$. Since $\xi_0\bar{\xi}_0=1$ and $\bar{\xi}_0=-\xi_0$, we have $\eta\bar{\eta}=1$ and $\mu=(\beta-\xi_0)^{-1}\lambda\eta$. Let

$$H_4=n_1H_2n_1=\{h_1(\lambda)h_2(\lambda)\mid\lambda\in F_0{}^*\},\quad J=\{h_1(\eta)\mid\eta\in F^*,\eta\bar{\eta}=1\}.$$

Then the above results yield that $H_0=H_4J$ and $T=H_0\cup\mathfrak{J}=T_0J$. Since J normalizes U_0 and centralizes both T_0 and $n_1n_2n_1$, the subgroup D_1 is a normal subgroup of C_1. And we have $C_1=D_1J$ and $D_1\cap J=H_4\cap J=\langle h_2(-1)\rangle$, where $h_2(-1)=h_1(\xi_0)$.

Finally we can show that there exists an isomorphism of D_1 onto $SU_3(q)$ which sends

$$x_1(\alpha)x_3(-\xi_0\alpha)x_4(\delta),\quad h_1(\lambda)h_2(\lambda),\quad x_2(\beta)h_1(\beta-\xi_0)^{-1}n_2x_2(\beta),\quad n_1n_2n_1$$

into

$$\begin{pmatrix}1 & 2\alpha & -\xi_0\alpha\bar{\alpha}+\delta\\ & 1 & -\xi_0\bar{\alpha}\\ & & 1\end{pmatrix},\ \begin{pmatrix}\lambda & & \\ & 1 & \\ & & \lambda^{-1}\end{pmatrix},\ \begin{pmatrix}(\beta-\xi_0)^{-1} & & \\ & \dfrac{\beta-\xi_0}{\beta+\xi_0} & \\ & & \beta+\xi_0\end{pmatrix},\ \begin{pmatrix} & & -1\\ & 1 & \\ 1 & & \end{pmatrix},$$

respectively.

This completes the proof of (5.2).

(5.3) *If* $q\equiv3 \pmod 8$, *then there is exactly one conjugacy classes of involutions in* $U_4(q)$ *with representative* $h_2(-1)$.

If $q\equiv-1 \pmod 8$, *then there exactly two conjugacy classes of involutions in* $U_4(q)$ *with representatives* $h_2(-1)$ *and* $h_1(\xi)n_2$.

The above result is known (*cf*. [4]). Thus we will sketch the proof.

Assume that $q\equiv-1 \pmod 4$ and let 2^m be the highest power of 2 dividing $q+1$, that is, $2^m\|q+1$. Then

$$m\geq2,\ 2\|\,|F_0{}^*|,\ 2^{m+1}\|\,|F^*|,\ 2^{m+3}\|\,|N|,\ 2^{3m+1}\|\,|C|,\ 2^{3m+1}\|\,|U_4(q)|.$$

Hence any Sylow 2-subgroup of C is a Sylow 2-subgroup of $U_4(q)$.

Thus any involution of $U_4(q)$ is conjugate in $U_4(q)$ to an involution of C. The following is the list of all involutions in C.

(I)

$h_2(-1)$;

$x_2(\beta)x_4(\mu\bar{\mu}\beta)h_1(\mu)n_1$, $\beta\in F_0$, $\mu\in F^*$;

$x_2(-\beta)x_4(-\delta)h_1(\mu)h_2(\mu\bar{\mu})n_2n_1n_2x_2(\beta)x_4(\delta)$, $\beta, \delta\in F_0$, $\mu\in F^*$;

$x_2(-\beta)x_4(-\delta)h_1(\mu)h_2(\lambda)(n_1n_2)^2x_2(\beta)x_4(\delta)$, $\beta, \delta\in F_0$, $\mu\in F^*$, $\lambda\in F_0{}^*$, $\bar{\mu}=\pm\mu$.

(II) only when $m\geq 3$ (ξ is a primitive 8-*th* root in F^*)

$x_2(-\beta)h_1(\xi)h_2(\lambda)n_2x_2(\beta)$, $\beta\in F_0, \lambda\in F_0{}^*$;

$x_4(-\delta)h_1(\xi\lambda)h_2(\lambda)n_1n_2n_1x_4(\delta)$, $\delta\in F_0, \lambda\in F_0{}^*$.

Note that

$$m=2 \Longleftrightarrow q\equiv 3 \pmod 8; \quad m\geq 3 \Longleftrightarrow q\equiv -1 \pmod 8).$$

Now we can show that every involution in (I) is conjugate in $U_4(q)$ to $h_2(-1)$, and every involution in (II) is conjugate in $U_4(q)$ to $h_1(\xi)n_2$.

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