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CERTAIN CLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS WITH FIXED FINITELY MANY COEFFICIENTS

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1. Introduction

Let S denote the class of the functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic and univalent in the unit disk $U=\{z:|z|<1\}$. A function $f(z)$ belonging to S is said to be starlike of order α if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\quad(z\in U) \tag{1.2}$$

for some $\alpha(0\leqq\alpha<1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Further, a function $f(z)$ belonging to S is said to be convex of order α if and only if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\quad(z\in U) \tag{1.3}$$

for some $\alpha(0\leqq\alpha<1)$. And we denote by $K(\alpha)$ the class of all convex functions of order α. Then $f(z)$ is in the class $K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$. We note that $S^*(\alpha)\subseteq S^*(0)\equiv S^*$ (S^* is the class of starlike functions), and $K(\alpha)\subseteq K(0)\equiv K$ (K is the class of convex functions).

The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [5], and were studied by Schild [6], MacGregor [2], Pinchuk [4], Jack [1] and others.

Let T be the subclass of S consisting of functions whose nonzero coefficients, from the second on, are negative. That is, an analytic and univalent functions $f(z)$ is in the class T if it can be expressed as

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n\quad(a_n\geqq0). \tag{1.4}$$

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We denote by $T^*(\alpha)$ and $C(\alpha)$ the classes obtained by taking intersections, respectively, of the classes $S^*(\alpha)$ and $K(\alpha)$ with T, that is, $T^*(\alpha)=T\cap S^*(\alpha)$ and $C(\alpha)=T\cap K(\alpha)$.

The classes $T^*(\alpha)$ and $C(\alpha)$ were studied by Silverman [9]. Shild [7] considered a subclass of T consisting of polynominals having $|z|=1$ as radius of univalence, Silverman [9] proved coefficients inequalities, distortion theorems, and covering theorems for $T^*(\alpha)$ and $C(\alpha)$, and Schild and Silverman [8] gave some interesting results for modified convolution product of functions belonging to the classes $T^*(\alpha)$ and $C(\alpha)$.

We request the results due to Silverman [9].

LEMMA 1. *Let the function $f(z)$ be defined by* (1.4). *Then $f(z)$ is in the class $T^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n\leqq 1-\alpha. \tag{1.5}$$

LEMMA 2. *Let the function $f(z)$ be defined by* (1.4). *Then $f(z)$ is in the class $C(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}n(n-\alpha)a_n\leqq 1-\alpha. \tag{1.6}$$

In view of the above lemmas, we observe that the function $f(z)$ in $T^*(\alpha)$ satisfies

$$a_n\leqq\frac{1-\alpha}{n-\alpha}\quad(n\geqq 2), \tag{1.7}$$

and the function $f(z)$ in $C(\alpha)$ satisfies

$$a_n\leqq\frac{1-\alpha}{n(n-\alpha)}\quad(n\geqq 2). \tag{1.8}$$

We introduce the following two classes to derive our results of great interest.

Let $T^*(\alpha, p_k)$ be the subclass of $T^*(\alpha)$ consisting of functions of the form

$$f(z)=z-\sum_{i=2}^{k}\frac{p_i(1-\alpha)}{i-\alpha}z^i-\sum_{n=k+1}^{\infty}a_nz^n\quad(a_n\geqq 0), \tag{1.9}$$

where $0\leqq p_i\leqq 1$ and $0\leqq\sum_{i=2}^{k}p_i\leqq 1$. Also let $C(\alpha, p_k)$ be the subclass of $C(\alpha)$ consisting of functions of the form

$$f(z)=z-\sum_{i=2}^{k}\frac{p_i(1-\alpha)}{i(i-\alpha)}z^i-\sum_{n=k+1}^{\infty}a_nz^n\quad(a_n\geqq 0), \tag{1.10}$$

where $0 \leqq p_i \leqq 1$ and $0 \leqq \sum_{i=2}^{k} p_i \leqq 1$. For $k=2$, the classes $T^*(\alpha, p_2)$ and $C(\alpha, p_2)$ were introduced by Silverman and Silvia [10], and Owa [3] proved some interesting theorems for modified convolution product of functions in $T^*(\alpha, p_2)$ and $C(\alpha, p_2)$. Furthermore, Silverman and Silvia [10] gave the extreme points of $T^*(\alpha, p_k)$ and $C(\alpha, p_k)$.

In the present paper, we prove some interesting results for modified convolution product of functions $f(z)$ in $T^*(\alpha, p_k)$ and $C(\alpha, p_k)$ and determine the radius of convexity for functions $f(z)$ belonging to the class $T^*(\alpha, p_k)$. Further, a theorem for integral operator $J(f)$ of $f(z)$ in $C(\alpha, p_k)$ is derived.

2. Modified convolution product

We begin with the proof of the following theorem we need to prove our results afterwards.

THEOREM 1. *Let the function $f(z)$ be defined by* (1.9). *Then $f(z)$ is in the class $T^*(\alpha, p_k)$ if and only if*

$$\sum_{n=k+1}^{\infty} (n-\alpha) a_n \leqq (1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right), \tag{2.1}$$

where $0 \leqq p_i \leqq 1$ and $0 \leqq \sum_{i=2}^{k} p_i \leqq 1$. The result (2.1) *is sharp.*

Proof. Putting

$$a_i = \frac{p_i(1-\alpha)}{i-\alpha} \quad (i=2, 3, \dots, k) \tag{2.2}$$

in Lemma 1, we have

$$\sum_{i=2}^{k} p_i (1-\alpha) + \sum_{n=k+1}^{\infty} (n-\alpha) a_n \leqq 1-\alpha \tag{2.3}$$

which implies (2.1). Further, taking the function $f(z)$ of the form

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i(1-\alpha)}{i-\alpha} z_i - \frac{(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{n-\alpha} z^n \tag{2.4}$$

for $n \geqq k+1$, we can see that the result (2.1) is sharp.

COROLLARY 1. *Let the function $f(z)$ defined by* (1.9) *be in the class $T^*(\alpha, p_i)$. Then*

$$a_n \leqq \left(\frac{1-\alpha}{n-\alpha}\right)\left(1-\sum_{i=2}^{k} p_i\right) \quad (n \geqq k+1). \tag{2.5}$$

Equality holds for the function $f(z)$ *given by* (2.4).

In the same manner, we have the following results for functions belonging to the class $C(\alpha, p_k)$.

THEOREM 2. *Let the function* $f(z)$ *be defined by* (1.10). *Then* $f(z)$ *is in the class* $C(\alpha, p_k)$ *if and only if*

$$\sum_{n=k+1}^{\infty} n(n-\alpha)a_n \leqq (1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right), \tag{2.6}$$

where $0 \leqq p_i \leqq 1$ *and* $0 \leqq \sum_{i=2}^{k} p_i \leqq 1$. *The result* (2.6) *is sharp for the function of the form*

$$f(z) = z - \sum_{i=2}^{k} \frac{p_i(1-\alpha)}{i(i-\alpha)} z^i - \frac{(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{n(n-\alpha)} z^n \tag{2.7}$$

for $n \geqq k+1$.

COROLLARY 2. *Let the function* $f(z)$ *defined by* (1.10) *be in the class* $C(\alpha, p_k)$. *Then*

$$a_n \leqq \left\{\frac{1-\alpha}{n(n-\alpha)}\right\}\left(1-\sum_{n=2}^{\infty} p_i\right) \quad (n \geqq k+1). \tag{2.8}$$

Equality hold for the function $f(z)$ *given by* (2.7).

Let $f * g(z)$ denote the modified convolution product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.4) and $g(z)$ given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geqq 0), \tag{2.9}$$

then $f * g(z)$ is defied by

$$f * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n. \tag{2.10}$$

By using Theorem 1 and Theorem 2, we prove the following interesting theorems for modified convolution product of functions belonging to $T^*(\alpha, p_k)$ and $C(\alpha, p_k)$.

THEOREM 3. *Let the function* $f(z)$ *defined by* (1.9) *be in the class* $T^*(\alpha, p_k)$. *Then* $f * f(z)$ *is in the class* $T^*(\alpha_1, q_k)$, *where* $\alpha_1 = \alpha(2-\alpha)$ *and*

$$q_k = \frac{p_k^2(k-2\alpha+\alpha^2)}{(k-\alpha)^2}. \tag{2.11}$$

Proof. It is clear that $0\leqq\alpha_1<1$, $0\leqq q_i\leqq 1$ and $0\leqq\sum_{i=2}^{k} q_i\leqq 1$. The definition of modified convolution product gives

$$f*f(z)=z-\sum_{i=2}^{k}\frac{p_i^2(1-\alpha)^2}{(i-\alpha)^2}z^i-\sum_{n=k+1}^{\infty}a_n^2z^n. \tag{2.12}$$

By virtue of Theorem 1, it is sufficient to show that

$$\sum_{n=k+1}^{\infty}(n-\alpha_1)a_n\leqq(1-\alpha_1)\left(1-\sum_{i=2}^{k}q_i\right). \tag{2.13}$$

Since $f(z)\in T^*(\alpha, p_k)$, we obtain that

$$\begin{aligned}\sum_{n=k+1}^{\infty}\{n-\alpha(2-\alpha)\}a_n^2&\leqq\sum_{n=k+1}^{\infty}(n-\alpha)a_n^2\\&\leqq(1-\alpha)^2\frac{\left(1-\sum_{i=2}^{k}p_i\right)^2}{k+1-\alpha}\\&=(1-\alpha_1)^2\frac{\left(1-\sum_{i=2}^{k}p_i\right)^2}{k+1-\alpha}.\end{aligned} \tag{2.14}$$

Therefore we only need to prove that

$$\left(1-\sum_{i=2}^{k}q_i\right)-\frac{\left(1-\sum_{i=2}^{k}p_i\right)^2}{k+1-\alpha}\geqq 0 \tag{2.15}$$

provided that $0\leqq p_i\leqq 1$ and $0\leqq\sum_{i=2}^{k}p_i\leqq 1$.

In fact, we have

$$\begin{aligned}&\left(1-\sum_{i=2}^{k}q_i\right)-\frac{\left(1-\sum_{i=2}^{k}p_i\right)^2}{k+1-\alpha}\\&=\frac{1}{k+1-\alpha}\left[(k-\alpha)\left\{1-\sum_{i=2}^{k}\frac{p_i^2(i-2\alpha+\alpha^2)}{(i-\alpha)^2}\right\}\right.\\&\left.+\left\{\sum_{i=2}^{k}p_i-\sum_{i=2}^{k}\frac{p_i^2(i-2\alpha+\alpha^2)}{(i-\alpha)^2}\right\}+\sum_{i=2}^{k}p_i\left(1-\sum_{i=2}^{k}p_i\right)\right]\\&\geqq 0\end{aligned} \tag{2.16}$$

which proves $f*f(z)\in T^*(\alpha_1, q_k)$. Thus we complete the proof of Theorem 3.

In the same manner, for the class $C(\alpha, p_k)$, we have

THEOREM 4. *Let the function $f(z)$ defined by (1.10) be in the class $C(\alpha, p_k)$. Then $f*f(z)$ is in the class $C(\alpha_1, q_k)$, where $\alpha_1=\alpha(2-\alpha)$ and*

$$q_k=\frac{p_k{}^2(k-2\alpha+\alpha^2)}{k^2(k-\alpha)^2}. \tag{2.17}$$

3. Radius of convexity

We determine the radius of convexity for functions belonging to the class $T^*(\alpha, p_k)$

THEOREM 5. *Let the function $f(z)$ defined by* (1.9) *be in the class $T^*(\alpha, p_k)$. Then $f(z)$ is convex in the disk $|z|<r_1$, where r_1 is the largest value for which*

$$\sum_{i=2}^{k}\frac{i^2p_i(1-\alpha)}{i-\alpha}r^{i-1}+\frac{n^2(1-\alpha)\left(1-\sum_{i=2}^{k}p_i\right)}{n-\alpha}r^{n-1}\leqq 1 \tag{3.1}$$

for $n\geqq k+1$. The result is sharp.

Proof. It suffices to prove that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq\frac{\sum_{i=2}\frac{i(i-1)p_i(1-\alpha)}{i-\alpha}r^{i-1}+\sum_{n=k+1}^{\infty}n(n-1)a_nr^{n-1}}{1-\sum_{i=2}^{k}\frac{ip_i(1-\alpha)}{i-\alpha}r^{i-1}-\sum_{n=k+1}^{\infty}na_nr^{n-1}} \leqq 1 \tag{3.2}$$

for $|z|\leqq r$. We denote that (3.2) holds if and only if

$$\sum_{i=2}^{k}\frac{i^2p_i(1-\alpha)}{i-\alpha}r^{i-1}+\sum_{n=k+1}^{\infty}n^2a_nr^{n-1}\leqq 1. \tag{3.3}$$

Since $f(z)\in T^*(\alpha, p_k)$, we may set

$$a_n=\frac{(1-\alpha)\left(1-\sum_{i=2}^{k}p_i\right)}{n-\alpha}\lambda_n \quad (n\geqq k+1), \tag{3.4}$$

where $\lambda_n\geqq 0\,(n\geqq k+1)$ and

$$\sum_{n=k+1}^{k}\lambda_n\leqq 1. \tag{3.5}$$

For each fixed r, we choose the integer $n_0=n_0(r)$ for which $n^2r^{n-1}/(n-\alpha)$ is maximal. Then

$$\sum_{n=k+1}^{\infty}n^2a_nr^{n-1}\leqq\frac{n_0{}^2(1-\alpha)\left(1-\sum_{i=2}^{k}p_i\right)}{n_0-\alpha}r^{n_0-1}. \tag{3.6}$$

Consequently, $f(z)$ is convex in $|z| \leqq r_1$ if

$$(3.7) \qquad \sum_{i=2}^{k} \frac{i^2 p_i(1-\alpha)}{i-\alpha} r^{i-1} + \frac{n_0{}^2(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{n_0-\alpha} r^{n_0-1} \leqq 1.$$

We find the value r_0 and the corresponding $n_0(r_0)$ so that

$$(3.8) \qquad \sum_{i=2}^{k} \frac{i^2 p_i(1-\alpha)}{i-\alpha} r_0{}^{i-1} + \frac{n_0{}^2(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{n_0-\alpha} r_0{}^{n_0-1} = 1.$$

Then this value r_0 is the radius of convexity of functions $f(z)$ in $T^*(\alpha, p_k)$. Furthermore, we prove that the result of the theorem is sharp for functions $f(z)$ given by (2.4).

4. Integral operator $J(f)$

We shall introduce the integral operator $J(f)$ defined by

$$(4.1) \qquad J(f) = \int_0^z \frac{f(t)}{t} dt.$$

Then it is well-known that $J(f)$ maps the class of starlike functions to the class of convex functions.

Hence $J(f) \in C(\alpha, p_k)$ if $f(z) \in T^*(\alpha, p_k)$.

THEOREM 6. *Let the function $f(z)$ defined by* (1.10) *be in the class $C(\alpha, p_k)$. Then $J(f)$ belong to the class $C(\alpha, q_k)$, where $q_k = p_k/k$.*

Proof. We note that

$$(4.2) \qquad J(f) = z - \sum_{i=2}^{k} \frac{(1-\alpha) q_i}{i(i-\alpha)} z^i - \sum_{n=k+1}^{\infty} \frac{a_n}{n} z^n.$$

Since

$$(4.3) \qquad \sum_{n=k+1}^{\infty} (n-\alpha) a_n \leqq \frac{(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{k+1}$$

for $f(z) \in C(\alpha, p_k)$, we obtain that

$$(4.4) \qquad \sum_{n=k+1}^{\infty} n(n-\alpha)\left(\frac{a_n}{n}\right) \leqq \frac{(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{k+1} \leqq (1-\alpha)\left(1-\sum_{i=2}^{k} \frac{p_i}{i}\right)$$

which completes the proof of the theorem.

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