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COUNTABLE COMPACTNESS, l.s.c. FUNCTIONS, AND FIXED POINTS

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In our previous works [13, 14, 15] we showed that certain maximum principles were formulated equivalently to fixed point results, and obtained their applications. In the present paper, such formulations are applied to characterize countably compact topological spaces. Consequently, some new fixed point results are obtained, and some of them include most of the extensions of the Furi-Vignoli type fixed point theorems on densifying maps.

Our tool is the following in [14, 15]. We add its simplified proof for the completeness.

THEOREM 1. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x, y \in X$. Then the following are equivalent:*

(i) *There exists an element $v \in A$ such that $G(v,w)$ for any $w \in X \setminus \{v\}$.*

(ii) *If $T: A \to 2^X$ is a multimap such that for any $x \in A \setminus T(x)$ there exists a $y \in X \setminus \{x\}$ satisfying $\sim G(x,y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f: A \to X$ is a map such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X \setminus \{x\}$ satisfying $\sim G(x,y)$, then f has a fixed element $v \in A$, that is, $v = fv$.*

(iv) *If $T: A \to 2^X \setminus \{\phi\}$ is a multimap such that $\sim G(x,y)$ holds for any $x \in A$ and any $y \in T(x) \setminus \{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

(v) *If $\mathfrak{F}$ is a family of maps $f: A \to X$ satisfying $\sim G(x,fx)$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

Here, $\sim$ denotes the negation.

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Proof. (i)$\Rightarrow$(ii). Suppose $v \notin T(v)$. Then there exists a $y \in X \setminus \{v\}$ satisfying $\sim G(v, y)$.

(ii)$\Rightarrow$(iii). Clear.

(iii)$\Rightarrow$(iv). Suppose T has no stationary element, that is, $T(x) \setminus \{x\} \neq \phi$ for any $x \in A$. Choose a choice function f on $\{T(x) \setminus \{x\} \mid x \in A\}$. Then f has no fixed element by its definition. However, for any $x \in A$, we have $x \neq fx$ and there exists a $y \in T(x) \setminus \{x\}$ satisfying $\sim G(x, y)$. Therefore, by (iii), f has a fixed element, a contradiction.

(iv)$\Rightarrow$(v). Define a multimap $T: A \to 2^X$ by $T(x) := \{fx \mid f \in \mathfrak{F}\} \neq \phi$ for all $x \in A$. Since $\sim G(x, fx)$ for any $x \in A$ and any $f \in \mathfrak{F}$, by (iv), T has a stationary element $v \in A$, which is a common fixed element of $\mathfrak{F}$.

(v)$\Rightarrow$(i). Suppose that for any $x \in A$, there exists a $y \in X \setminus \{x\}$ satisfying $\sim G(x, y)$. Choose fx to be one of such y. Then $f: A \to X$ has no fixed element by its definition. However, $\sim G(x, fx)$ for all $x \in A$. Let $\mathfrak{F} = \{f\}$. By (v), f has a fixed element, a contradiction.

In [13, 14, 15], Theorem 1 is applied to Ekeland's variational principle, Zorn's lemma, and other maximum principles.

The following is a simple consequence of Theorem 1:

COROLLARY ([8, 11]). *Let f be a selfmap of a set X such that the function $x \mapsto d(fx, f^2x)$, $x \in X$, has a minimum value at some $a \in X$, where d is a nonnegative real-valued function on $X \times X$. Suppose that, for all $x, y \in X$ with $fx \neq fy$, there is a map $g: X \to fX$ which commutes with f and satisfies*

$$d(gx, gy) < d(fx, fy).$$

Then f has a fixed element.

Proof. From the hypothesis, we have

(i) $d(fa, f^2a) \leq d(fx, f^2x)$ for any $x \in X \setminus \{a\}$.

Therefore, by Theorem 1(iii), the conclusion follows. In fact, since $f: fX \to X$, for any $x \in X$ satisfying $fx \neq f^2x$, there is a $gx \in fX \setminus \{fx\}$ satisfying

$$d(gx, gfx) < d(fx, f^2x).$$

Hence, by Theorem 1(iii), f has a fixed element. Furthermore, from the proof of Theorem 1, fa is a fixed element of f.

A slight generalization of Corollary and its applications can be found in Jungck [8].

For countably compact spaces, we have the following:

THEOREM 2. *Let X be a countably compact space and $F: X \to \boldsymbol{R}$ a real-valued l. s. c. function. Then the following equivalent conditions hold*:

(i) *F attains its infimum on X, that is, there exists a $v \in X$ such that $F(v) \leq F(w)$ for all $w \neq v$.*

(ii) *If $T: X \to 2^X$ is a multimap such that for any $x \in X \setminus T(x)$, there exists a $y \in X$ satisfying $F(x) > F(y)$, then T has a fixed point.*

(iii) *If $f: X \to X$ is a map such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X$ satisfying $F(x) > F(y)$, then f has a fixed point.*

(iv) *If $T: X \to 2^X \setminus \{\phi\}$ is a multimap such that $F(x) > F(y)$ holds for any $x \in X$ and any $y \in T(x) \setminus \{x\}$, then T has a stationary point.*

(v) *If $\mathfrak{F}$ is a family of selfmaps f of X satisfying $F(x) > F(fx)$ for all $x \in X$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed point.*

REMARK. Theorem 2(i) is well-known ([1, 2]), and 2(iii) is due to C. S. Wong [19] and generalizes results of Edelstein [4] and J. S. W. Wong [20]. Note also that the hypothesis of 2(iii) simply tells us that the infimum point of F is fixed under f.

Now, we obtain the following characterizations of countably compact spaces:

THEOREM 3. *Let X be a topological space. Then the following are equivalent*:

(0) *X is countably compact.*

(i) *Every real-valued l. s. c. function F on X attains its infimum.*

(iii) *For any real-valued l. s. c. function F on X and any map $f: X \to X$ such that for any $x \in X$ with $x = fx$ there exists a $y \in X$ satisfying $F(x) > F(y)$, f has a fixed point.*

REMARK (0)$\Leftrightarrow$(i) is due to Blatter [1] and C. S. Wong [18]. By imitating Theorems 1 and 2, we can add conditions (ii), (iv), and (v) to Theorem 3. Note that (i)$\sim$(v) are all equivalent by Theorems 1 and 2.

The following is a variation of Theorem 3.

THEOREM 4. *Let A be a closed subset of a topological space X. Then the following are equivalent*:

(0) *A is countably compact.*

(i) *For any l.s.c. function $F: X \to [0, \infty)$ such that* $\inf F(A)=0$, *we have $F(v)=0$ for some $v \in A$.*

(iii) *For any l.s.c. function $F: X \to [0, \infty)$ such that* $\inf F(A)=0$, *and for any map $f: A \to X$ such that $x \neq fx$ implies the existence of a $y \in X$ with $F(x) > F(y)$, f has a fixed point.*

REMARK. (0)$\Leftrightarrow$(i) is due to C. S. Wong [19]. We can also have equivalent formulations (ii), (iv), and (v).

Theorems 3 and 4 can be applied to a compact L-space since every real-valued l. s. c. function defined on such L-space attains its infimum [9]. Therefore, the following fixed point theorem follows:

THEOREM 5(iii). *Let $(X, \to)$ be a compact L-space, $f: X \to X$, and $d: X \times X \to [0, \infty)$ such that the function $x \mapsto d(x, fx)$, $x \in X$, is l. s. c. If for any $x \in X$ with $x \neq fx$, there exists a $y \in X \setminus \{x\}$ such that*

$$d(y, fy) < d(x, fx),$$

then f has a fixed point.

REMARK. Theorem 5(i), (ii), (iv), and (v) can be stated. Note that Theorem 5(iii) is essentially due to Kasahara [9]. A number of consequences and variations of Theorem 5(iii) are given in [9]. For example, in case where $y = fx$ in Theorem 5(iii), we have a generalization of Edelstein's theorem [4].

An application of Theorem 5(iii) is the following Furi-Vignoli type fixed point theorem:

COROLLARY. *Let (X, d) be a complete metric space and F a real-valued l. s. c. function defined on $X \times X$. Let f be a continuous densifying selfmap of X such that for any $x \in X$ there exists a positive integer $n(x)$ such that if $x \neq fx$ then*

$$F(f^n x, f^{n+1} x) < F(x, fx)$$

holds. If the orbit $O(x) = \{f^i x \mid i \in \omega\}$ of an $x \in X$ is bounded, then f has a fixed point.

Proof. As in [6], $\bar{O}(x)$ is compact. Note that $y \mapsto F(y, fy)$, $y \in \bar{O}(x)$, is l. s. c. Therefore, Corollary follows from Theorem 5(iii).

REMARK. Most of the Furi–Vignoli type fixed point theorems appeared in Furi–Vignoli [5], Daneš [3], Iséki [6, 7], Khan–Singh [10], Park [12], Singh–Zorzitto [16], and Thomas [17] are consequences of Theorem 5(iii) and Corollary.

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