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HYPERSURFACES WITH PARTIALLY INTEGRABLE STRUCTURE OF AN ODD-DIMENSIONAL SPHERE

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§ 0. Introduction

D. E. Blair [1], S. Ishihara ([4], [14]). U-H. Ki ([4], [7], [16]), G. D. Ludden [1], M. Okumura [20], J. S. Pak [7] and K. Yano ([1], [14], [16], [18]) studied a structure induced on a hypersurface of an almost contact manifold or a submanifold of codimension 2 of an almost complex manifold. When the ambient manifold admits a Riemannian metric, Yano and Okumura called the structure the (f, g, u, v, λ)-structure ([20]).

As is well known, the odd-dimensional sphere $S^{2n+1}(1)$ of radius 1 has an almost contact metric structure induced from the natural Kaehlerian structure of an even-dimensional Euclidean space and consequently a hypersurface immersed in $S^{2n+1}(1)$ admits the so-called (f, g, u, v, λ)-structure.

On the other hand, H. Suzuki [11] investigated the integrability conditions of an almost complex structure F constructed from the (f, g, u, v, λ)-structure. Using local components of the Nijenhuis tensor formed with F, many authors ([1], [3], [5], [6], [8], [12], [13], [14] etc.) studied hypersurfaces of $S^{2n+1}(1)$.

In the present paper, we characterize hypersurfaces of an odd-dimensional unit sphere under one of the integrable conditions of F above.

In § 1, we recall fundamental properties and structure equations for hypersurfaces immersed in a Sasakian manifold and introduce Yano-Kon's work (see THEOREM A and B).

§ 2 is devoted to define the partially integrability of a manifold with (f, g, u, v, λ)-structure and to find some useful lemmas on the hypersurface with such a structure of $S^{2n+1}(1)$.

In § 3, we study complete hypersurfaces with partially integrable

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(f, g, u, v, λ)-structure immersed in $S^{2n+1}(1)$ and characterize the hypersurfaces under certain conditions (see THEOREM 8~11).

In the last §4, we prove that if a pseudo-Einstein hypersurface of $S^{2n+1}(1)$ admits partially integrable structure, then the Sasakian structure vector is tangent to the hypersurface.

§1. Hypersurfaces of a Sasakian manifold

Let N be a $(2n+1)$-dimensional Sasakian manifold with structure tensor (ϕ, G, ξ). Then the structure tensor of N satisfies [9]

$$(1.1)\qquad \begin{cases} \phi^2 X = -X + \eta(X)\xi, \\ \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y) \end{cases}$$

for any vector fields X and Y on N, where the 1-form η is defined by $\eta(X) = G(X, \xi)$ for any vector field X on N. We donote by ∇ the operator of covariant differentiation with respect to the metric tensor G on N. We then have

$$(1.2)\qquad \nabla_X \xi = \phi X, \quad (\nabla_X \phi) Y = -G(X, Y)\xi + \eta(X) Y$$

for any vector fields X and Y on N.

Let M be a $2n$-dimensional orientable and connected real hypersurface in N covered by a system of coordinate neighborhoods $\{W; y^a\}$, that is, M be isometrically immersed in N by the immersion $i: M \to N$ as a hypersurface. (Throughout this paper the indices a, b, c, d and e run over the range $\{1, 2, \cdots, 2n\}$).

Denoting by B=di, the differential of i, then the induced Riemannian metric g on M is given by

$$(1.3)\qquad G(BX, BY) = g(X, Y)$$

for any vector fields X and Y tangent to M because the immersion is isometric.

Since M is orientable, we can choose a unit normal C to M globally along M. Thus, for any vector field X tangent to M, we can put

$$(1.4)\qquad \phi BX = BfX - u(X)C,$$

$$(1.5)\qquad \phi C = BU,$$

$$(1.6)\qquad \xi = BV + \lambda C,$$

where we have put $u(X) = g(U, X)$, f being a tensor field of type

(1, 1), U and V vector fields and λ a function on M.

Applying ϕ to both sides of the equations (1.4)～(1.6) and using (1.1) and (1.3), we find ([1], [3], [5] etc.)

$$(1.7)\qquad \begin{cases} f_c{}^e f_e{}^a = -\delta_c{}^a + u_c u^a + v_c v^a, \\ u_e f_c{}^e = -\lambda v_c, \quad v_e f_c{}^e = \lambda u_c, \\ f_e{}^a u^e = \lambda v^a, \quad f_e{}^a v^e = -\lambda u^a, \\ u_e v^e = 0, \quad u_e u^e = v_e v^e = 1 - \lambda^2, \\ g_{ed} f_c{}^e f_b{}^d = g_{cb} - u_c u_b - v_c v_b, \end{cases}$$

where $f_c{}^a$, g_{cb}, u^a and v^a are components of f, g, U and V respectively and u_a and v_a are associated 1-forms of u^a and v^a respectively. Therefore M admits the so-called (f, g, u, v, λ)-structure ([16], [18]). If we put $f_{cb} = f_c{}^a g_{ab}$, then we can easily see that f_{cb} is skew-symmetric.

It is well known that the (f, g, u, v, λ)-structure induced on the hypersurface M of a Sasakian manifold N with the second fundamental form H satisfies ([1], [3], [5] etc.)

$$(1.8)\qquad \nabla_c f_b{}^a = -g_{cb} v^a + \delta_c{}^a v_b + h_{cb} u^a - h_c{}^a u_b,$$

$$(1.9)\qquad \nabla_c u_b = \lambda g_{cb} + h_{ce} f_b{}^e,$$

$$(1.10)\qquad \nabla_c v_b = f_{cb} + \lambda h_{cb},$$

$$(1.11)\qquad \nabla_c \lambda = -u_c - h_{ce} v^e,$$

where h_{cb} are components of H, $h_c{}^a = h_{cb} g^{ab}$, $(g^{cb}) = (g_{cb})^{-1}$ and ∇_c being the operator of the covariant differentiation with respect to g_{cb}.

We note from (1.10) that the function $1-\lambda^2$ is nonzero almost everywhere on M because h_{cb} is symmetric and f_{cb} is skew-symmetric.

We now define a tensor field T of type (0, 2) defined by

$$(1.12)\qquad T_{cb} = f_c{}^e \nabla_e u_b - f_b{}^e \nabla_e u_c - (\nabla_c f_b{}^e - \nabla_b f_c{}^e) u_e - \lambda (\nabla_c v_b - \nabla_b v_c).$$

If T vanishes identically, then the (f, g, u, v, λ)-structure induced on M is said to be *partially integrable* (cf. [3]).

Substituting (1.8)～(1.10) into (1.12), we get

$$(1.13)\qquad T_{cb} = -u_c v_b + u_b v_c + (h_{ce} u^e) u_b - (h_{be} u^e) u_c.$$

When the ambient manifold is a unit sphere $S^{2n+1}(1)$, the equations of Gauss and Codazzi are given respectively by

$$(1.14)\qquad K_{dcb}{}^a = \delta_d{}^a g_{cb} - \delta_c{}^a g_{db} + h_d{}^a h_{cb} - h_c{}^a h_{db},$$

$$(1.15)\qquad \nabla_c h_{ba} - \nabla_b h_{ca} = 0,$$

$K_{dcb}{}^a$ being components of the curvature tensor of M. Denoting by K_{cb}

the components of the Ricci tensor S of the hypersurface, we have from (1.14)

$$(1.16) \qquad K_{cb}=(2n-1)g_{cb}+hh_{cb}-h_{ce}h_b{}^e,$$

where $h=h_{cb}g^{cb}$. Thus the scalar curvature K of M is written in the form

$$(1.17) \qquad K=2n(2n-1)+h^2-h_{cb}h^{cb}.$$

If $\nabla S=0$ on M, then the hypersurface M is said to be Ricci parallel.

Let M be a $2n$-dimensional hypersurface with $\lambda=0$ of $S^{2n+1}(1)$. If the Ricci tensor S of M is of the form

$$(1.18) \qquad S(\phi^2X, \phi^2Y)=ag(\phi^2X, \phi^2Y)+bu(\phi^2X)u(\phi^2Y)$$

for any vector fields X and Y tangent to M, a and b being constants, then M is called a *pseudo-Einstein hypersurface* of $S^{2n+1}(1)$ [17]. The equation (1.18) is equivalent to

$$(1.19) \qquad K_{cb}=a(g_{cb}-v_cv_b)+bu_cu_b+(K_{ce}v^e)v_b+(K_{be}v^e)v_c \\ -(K_{ed}v^ev^d)v_cv_b$$

with the aid of (1.4), (1.5) and (1.7) with $\lambda=0$.

Let C^{n+1} be the space of $(n+1)$-tuples of complex numbers $(z_1, \cdots, z_{n+1})$. Put $S^{2n+1}=\left\{(z_1, \cdots, z_{n+1})\in C^{n+1}: \sum_{j=1}^{n+1}|z_j|^2=1\right\}$. For a positive number r we denote by M_0 $(2n, r)$ a hypersurface of S^{2n+1} defined by

$$\sum_{j=1}^{n}|z_j|^2=r|z_{n+1}|^2, \qquad \sum_{j=1}^{n+1}|z_j|^2=1.$$

For an integer $m(2\leqq m\leqq n-1)$ a hypersurface $M(2m, m, (m-1)/(n-m))$ of S^{2n+1} is defined by

$$\sum_{j=1}^{m}|z_j|^2=\frac{m-1}{n-m}\sum_{j=m+1}^{n+1}|z_j|^2, \qquad \sum_{j=1}^{n+1}|z_j|^2=1.$$

For a number $t(0<t<1)$ we denote by $M(2n, t)$ a hypersurface of S^{2n+1} defined by

$$\sum_{j=1}^{n+1}|z_j{}^2|^2=t, \quad \sum_{j=1}^{n+1}|z_j|^2=1.$$

In the right of these facts, K. Yano and M. Kon proved the following two theorems.

THEOREM A ([17]). *If M is a complete pseudo-Einstein hypersurface with $\lambda=0$ in $S^{2n+1}(1)$ $(n\geqq 3)$, then M is congruent to some M_0 $(2n, r)$ or to some $M(2n, m, (m-1)/(n-m)$ or to $M(2n, 1/(n-1))$.*

THEOREM B ([17]). *If M is a complete pseudo-Einstein minimal hypersurface with $\lambda=0$ in $S^{2n+1}(1)$ $(n\geqq 3)$, then M is congruent to $M_0(2n, 2n-1)$ or to $M(2n, (n+1)/2, 1)$. In the latter case, n is odd.*

§ 2. Lemmas on partially integrable structure

From now on we suppose that the hypersurface M of an odd-dimensional unit sphere $S^{2n+1}(1)$ has the partially integrable (f, g, u, v, λ)-structure. Then we have from (1.13)

$$(h_{ce}u^e)u_b-(h_{be}u^e)u_c+u_bv_c-u_cv_b=0.$$

Transvecting u^b and using (1.7), we get

$$h_{ce}u^e=-v_c+\alpha u_c \tag{2.1}$$

because the function $1-\lambda^2$ does not vanish almost everywhere, where we have put

$$h_{cb}u^cu^b=\alpha(1-\lambda^2). \tag{2.2}$$

If we transvect (1.11) with u^c and make use of (1.7) and (2.1), then we have

$$u^e\nabla_e\lambda=0. \tag{2.3}$$

Transvecting (1.16) with u^b and taking account of (1.11), (2.1) and (2.2), we have

$$K_{ce}u^e=(2n-2+h\alpha-\alpha^2)u_c+(\alpha-h)v_c-\nabla_c\lambda. \tag{2.4}$$

First of all, we prove

LEMMA 1. *If M is a hypersurface of $S^{2n+1}(1)$ with partially integrable (f, g, u, v, λ)-structure, then the function λ is constant.*

Proof. Differentiating (2.1) covariantly along M, we find

$$(\nabla_ch_{be})u^e+h_b{}^e\nabla_cu_e=-\nabla_cv_b+(\nabla_c\alpha)u_b+\alpha\nabla_cu_b$$

Substituting (1.9) and (1.10) into the above equation, we have

$$(\nabla_ch_{be})u^e+2\lambda h_{cb}+h_b{}^eh_{ca}f_e{}^a=-f_{cb}+(\nabla_c\alpha)u_b+\alpha(\lambda g_{cb}+h_{ce}f_b{}^e),$$

from which we have

$$(2.5)\qquad 2h_{be}h_{ca}f^{ea}=2f_{bc}+(\nabla_c\alpha)u_b-(\nabla_b\alpha)u_c+\alpha(h_{ce}f_b{}^e-h_{be}f_c{}^e),$$

taking the skew-symmetric part and using (1.15).

Transvecting (2.5) with u^b and making use of (1.7), (1.11), (2.1) and (2.3), we find

$$(2.6)\qquad (1-\lambda^2)\nabla_c\alpha=\beta u_c-\alpha\lambda\nabla_c\lambda-\lambda(\alpha^2+4)v_c,$$

where we have put $\beta=u^e\nabla_e\alpha$.

Covariant differentiation of (2.6) gives

$$\begin{aligned}&(1-\lambda^2)\nabla_c\nabla_b\alpha-(\nabla_c\lambda^2)(\nabla_b\alpha)+\frac{1}{2}(\nabla_c\alpha)\nabla_b\lambda^2+\frac{1}{2}\alpha\nabla_c\nabla_b\lambda^2\\ &=(\nabla_c\beta)u_b+\beta(\lambda g_{cb}+h_{ce}f_b{}^e)-(\alpha^2+4)(\nabla_c\lambda)v_b-2\lambda\alpha(\nabla_c\alpha)v_b\\ &\quad-\lambda(\alpha^2+4)(f_{cb}+\lambda h_{cb})\end{aligned}$$

with the aid of (1.9) and (1.10), from which we have

$$\begin{aligned}(2.7)\qquad &3\lambda\{(\nabla_b\lambda)(\nabla_c\alpha)-(\nabla_c\lambda)(\nabla_b\alpha)\}\\ &=(\nabla_c\beta)u_b-(\nabla_b\beta)u_c+\beta(h_{ce}f_b{}^e-h_{be}f_c{}^e)-(\alpha^2+4)(v_b\nabla_c\lambda-v_c\nabla_b\lambda)\\ &\quad+2\alpha\lambda(v_c\nabla_b\alpha-v_b\nabla_c\alpha)-2\lambda(\alpha^2+4)f_{cb},\end{aligned}$$

taking the skew-symmetric part.

If we transvect (2.7) with u^b and take account of (1.7), (2.2) and (2.3), then we obtain

$$(2.8)\qquad (1-\lambda^2)\nabla_c\beta=(u^e\nabla_e\beta)u_c-2\lambda\beta\nabla_c\lambda-\lambda\{3\beta\alpha+2\lambda(\alpha^2+4)\}v_c.$$

Substituting (2.6) and (2.8) into (2.7), we get

$$\begin{aligned}(2.9)\qquad &\frac{\lambda\beta}{1-\lambda^2}(u_c\nabla_b\lambda-u_b\nabla_c\lambda)+\frac{8\lambda^2+\alpha^2+4}{1-\lambda^2}(v_b\nabla_c\lambda-v_c\nabla_b\lambda)\\ &=\frac{\lambda}{1-\lambda^2}\{\beta\alpha+2\lambda(\alpha^2+4)\}(u_cv_b-u_bv_c)+\beta(h_{ce}f_b{}^e-h_{be}f_c{}^e)\\ &\quad-2\lambda(\alpha^2+4)f_{cb}\end{aligned}$$

because the function $1-\lambda^2$ is nonzero almost everywhere.

Transvecting (2.9) with v^b and using (1.7), (2.1) and (2.3), we find

$$(8\lambda^2+\alpha^2+4)\nabla_c\lambda-\beta(\nabla_e\lambda)f_c{}^e=-\frac{\lambda\beta\gamma}{1-\lambda^2}u_c+\frac{\gamma(8\lambda^2+\alpha^2+4)}{1-\lambda^2}v_c,$$

where we have put $\gamma=v^e\nabla_e\lambda$.

Transvecting this equation with $f_b{}^c$ yields

$$(8\lambda^2+\alpha^2+4)f_b{}^e\nabla_e\lambda+\beta\nabla_b\lambda-\beta\gamma v_b=\frac{\lambda^2\beta\gamma}{1-\lambda^2}v_b+\frac{\lambda\gamma(8\lambda^2+\alpha^2+4)}{1-\lambda^2}u_b,$$

where we have used here (1.7) and (2.3). Thus the last two equations give

$$\nabla_b\lambda=\delta v_b \tag{2.10}$$

for some function δ on M. Differentiating this covariantly, we find

$$\nabla_c\nabla_b\lambda=(\nabla_c\delta)v_b+\delta(f_{cb}+\lambda h_{cb})$$

because of (1.10), from which we have

$$(\nabla_c\delta)v_b-(\nabla_b\delta)v_c+2\delta f_{cb}=0,$$

taking the skew-symmetric part.

Transvecting the above equation with v^b yields

$$(1-\lambda^2)\nabla_c\delta=(v^e\nabla_e\delta)v_c-2\lambda\delta u_c.$$

Therefore, the last two relationships imply

$$\frac{\lambda\delta}{1-\lambda^2}(u_cv_b-u_bv_c)-\delta f_{cb}=0.$$

If we transvect f^{cb} and use (1.7), we see that δ vanishes identically on M for $n>1$. Hence it follows from (2.10) that λ is constant on M. This completes the proof of the lemma.

LEMMA 2. *Under the same assumptions as those stated in* LEMMA 1, *if* $\alpha=h$, *then* $\lambda=0$ *on* M.

Proof. Since λ is constant and $\alpha=h$, (2.6) and (2.8) reduce respectively to

$$(1-\lambda^2)\nabla_c h=\beta u_c-\lambda(h^2+4)v_c, \tag{2.11}$$

$$(1-\lambda^2)\nabla_c\beta=(u^e\nabla_e\beta)u_c-\lambda\{3h\beta+2\lambda(h^2+4)\}v_c. \tag{2.12}$$

Differentiating (2.11) covariantly and substituting (1.9) and (1.10), we find

$$\begin{aligned}(1-\lambda^2)\nabla_c\nabla_b h=&(\nabla_c\beta)u_b+\beta(\lambda g_{cb}+h_{ce}f_b{}^e)-2\lambda h(\nabla_c h)v_b\\&-\lambda(h^2+4)(f_{cb}+\lambda h_{cb})\end{aligned}$$

because of the fact that $\lambda=$const., from which we have

(2.13) $$(\nabla_c\beta)u_b-(\nabla_b\beta)u_c+\beta(h_{ce}f_b{}^e-h_{be}f_c{}^e)-2\lambda h(v_b\nabla_c h-v_c\nabla_b h)$$
$$-2\lambda(h^2+4)f_{cb}=0,$$

taking the skew-symmetric part.

Substituting (2.11) and (2.12) into (2.13), we have

$$\beta(h_{ce}f_b{}^e-h_{be}f_c{}^e)-2\lambda(h^2+4)f_{cb}-\frac{\lambda\{h\beta+2\lambda(h^2+4)\}}{1-\lambda^2}(v_cu_b-v_bu_c)=0.$$

Thus, by transvecting f^{cb} and taking account of (1.7), (1.11) and (2.1), we see that λ vanishes on M for $n>1$. Therefore, the lemma is proved.

LEMMA 3. *Under the same assumptions as those stated in* LEMMA *1, if the mean curvature is constant, then we have* $\lambda=0$ *on* M.

Proof. Since λ is constant, (1.11) reduces to

(2.14) $$h_{ce}v^e=-u_c.$$

Operating ∇^c to (2.14) and making use of (1.9), (1.10) and (1.15), we get

$$(\nabla_e h)v^e+\lambda h_{cb}h^{cb}=-2n\lambda.$$

Since h is constant, it follows that λ vanishes identically on M. This completes the proof.

LEMMA 4. *Under the same assumptions as those stated in* LEMMA *1, if* $\lambda=0$, *then* α *is constant on* M.

Proof. Since $\lambda=0$ on M, (2.6), (2.8) and (2.9) reduce respectively to

(2.15) $$\nabla_c\alpha=\beta u_c,$$
(2.16) $$\nabla_c\beta=(u^e\nabla_e\beta)u_c,$$
(2.17) $$\beta(h_{ce}f_b{}^e-h_{be}f_c{}^e)=0.$$

Thus, (2.5) implies that

(2.18) $$\beta(h_{be}h_a{}^ef_c{}^a-f_{bc})=0$$

Now we put a set $A=\{x\in M\,|\,\beta(x)\neq 0\}$. Then A is open in M. Thereby at each point of A, we have from (2.17) and (2.18)

(2.19) $$h_{ce}f_b{}^e-h_{be}f_c{}^e=0,$$
(2.20) $$h_{be}h_a{}^ef_c{}^a-f_{bc}=0.$$

Transvecting (2. 20) with f^{cb} and using (1. 7) with $\lambda=0$, (2. 1) and (2. 14), we get on A

$$h_{cb}h^{cb}=\alpha^2+4-2n.$$

On the other hand, we have on A

$$(\nabla_c u_b)(\nabla^c u^b)=(h_{ce}f_b{}^e)(h_a{}^c f^{ba})=h_{cb}h^{cb}-\alpha^2-2$$

with the aid of (1. 7) and (1. 9) with $\lambda=0$, (2. 1) and (2. 14). Thus, it follows that

$$h_{ce}f_b{}^e=0 \tag{2.21}$$

on A. Transvecting (2. 21) with $f_a{}^b$ yields

$$h_{ca}=\alpha u_c u_a-(u_c v_a+u_a v_c)$$

on A because of (1. 7) with $\lambda=0$, (2. 1) and (2. 14).

Differentiating the above equation covariantly and taking account of (1. 12) with $\lambda=0$, (2. 15) and (2. 21), we get on A

$$\nabla_b h_{ca}=\beta u_b u_c u_a-u_c f_{ba}-u_a f_{bc},$$

from which we have

$$u_b f_{ca}-u_c f_{ba}-2u_a f_{bc}=0,$$

taking the skew-symmetric part with respect to b and c and using (1. 15). Thus, it follows that $f_{cb}=0$, which contradicts the fact that $n>1$. Hence the set A is void and so α is constant because of (2. 15). Therefore the lemma is proved.

Combining LEMMA 2∼4, we conclude

LEMMA 5. *Let M be a hypersurface with partially integrable (f, g, u, v, λ)-structure of $S^{2n+1}(1)$. If the mean curvature h of M is constant or h is equal to α, then λ vanishes identically and α is constant on M.*

LEMMA 6. *Under the same assumptions as those stated in LEMMA 1, we have*

$$\frac{1}{2}\|\nabla_c u_b+\nabla_b u_c\|^2=h_{cb}h^{cb}-2n-\alpha h+\lambda^2(4n+4+\alpha^2). \tag{2.22}$$

Proof. From the Ricci identity for u^a, that is,

$$\nabla_d\nabla_c u^a-\nabla_c\nabla_d u^a=K_{dcb}{}^a u^b,$$

we find

$$\nabla^b \nabla_c u_b = K_{cb} u^b$$

because of LEMMA 1 and (1.9). From this and (2.4) with λ=const., we have

$$u^c \nabla^b \nabla_c u_b = (1-\lambda^2)(2n-2+h\alpha-\alpha^2). \tag{2.23}$$

On the other hand, transvecting (1.9) with u^c and using (1.7) and (2.1), we get

$$u^c \nabla_c u_b = -\lambda \alpha v_b.$$

If we apply ∇^b to this and make use of the fact that λ is constant, we obtain

$$(\nabla_c u_b)(\nabla^b u^c) + u^c \nabla^b \nabla_c u_b = -\lambda\{(\nabla^b \alpha) v_b + \alpha \nabla^b v_b\}.$$

Taking account of (1.10), (2.6) with λ=const. and (2.23), we have

$$(\nabla_c u_b)(\nabla^b u^c) = \lambda^2(2n+2) - (2n-2+h\alpha-\alpha^2). \tag{2.24}$$

We have from (1.7), (1.9), (2.1) and (2.14)

$$\|\nabla_c u_b\|^2 = h_{cb} h^{cb} - \alpha^2 - 2 + \lambda^2(2n+2+\alpha^2). \tag{2.25}$$

Substituting (2.24) and (2.25) into the identity:

$$\frac{1}{2}\|\nabla_c u_b + \nabla_b u_c\|^2 = \|\nabla_c u_c\|^2 + (\nabla_c u_b)(\nabla^b u^c),$$

we obtain (2.22). Therefore LEMMA 6 is proved.

LEMMA 7. *Under the same assumptions as those stated in* LEMMA 1, *if λ vanishes identically and*

$$h_{cb} h^{cb} = h\alpha + 2n \tag{2.26}$$

holds, then we have

$$h_{ce} h_b{}^e = \alpha h_{cb} + g_{cb}. \tag{2.27}$$

Proof. From (2.22) and (2.26) we have

$$\nabla_c u_b + \nabla_b u_c = 0$$

because of the fact that λ vanishes. And consequently

$$h_{ce} f_b{}^e + h_{be} f_c{}^e = 0 \tag{2.28}$$

with the aid of (1.9). Since $\lambda=0$, (2.6) becomes

$$\nabla_c\alpha=\beta u_c. \tag{2.29}$$

If we take account of (2.28) and (2.29), then (2.5) reduces to

$$h_b{}^e h_{ea} f_c{}^a=f_{cb}+h_{be}f_c{}^e.$$

Transvecting this with $f_d{}^c$ and make use of (1.7), (2.1) and (2.14), we obtain (2.27). This completes the proof of LEMMA 7.

§3. Complete hypersurfaces of $S^{2n+1}(1)$ with partially integrable structure

First of all we prove

THEOREM 8. *Let M be a complete and connected hypersurface of an odd-dimensional sphere $S^{2n+1}(1)$ $(n>1)$. If the induced (f, g, u, v, λ)-structure is partially integrable and Ricci tensor S of M satisfies $(\nabla_X S)Y=(\nabla_Y S)X$ for any vector fields X and Y tangent to M, then M is a product of two spheres $S^p\times S^{2n-p}$, p being an odd number.*

Proof. From LEMMA 1, we know that λ is constant on M. Thus (2.4) reduces to

$$K_{be}u^e=(2n-2+h\alpha-\alpha^2)u_b+(\alpha-h)v_b. \tag{3.1}$$

Transvecting (1.16) with v^c and using (2.1) and (2.14), we find

$$K_{be}v^e=2(n-1)v_b+(\alpha-h)u_b. \tag{3.2}$$

Differentiating (3.2) covariantly and substituting (1.9)' and (1.10), we get

$$(\nabla_c K_{be})v^e+K_{be}(f_c{}^e+\lambda h_c{}^e)$$
$$=2(n-1)(f_{cb}+\lambda h_{cb})+\nabla_c(\alpha-h)u_b+(\alpha-h)(\lambda g_{cb}+h_{ce}f_b{}^e),$$

from which we have

$$K_{be}f_c{}^e-K_{ce}f_b{}^e=4(n-1)f_{cb}+(\alpha-h)(h_{ce}f_b{}^e-h_{be}f_c{}^e)$$
$$+\nabla_c(\alpha-h)u_b-\nabla_b(\alpha-h)u_c, \tag{3.3}$$

taking the skew-symmetric part and using (1.16) and the fact that $\nabla_c K_{be}=\nabla_b K_{ce}$.

Transvecting (3.3) with u^b and taking account of (1.7), (2.1), (2.14), (3.1) and (3.2), we find

(3.4) $$(1-\lambda)^2\nabla_c(\alpha-h)=\{u^e\nabla_e(\alpha-h)\}u_c.$$

Thus, (3.3) becomes

(3.5) $$K_{be}f_c{}^e-K_{ce}f_b{}^e=4(n-1)f_{cb}+(\alpha-h)(h_{ce}f_b{}^e-h_{be}f_c{}^e).$$

Transvecting (3.5) with f^{cb} and taking account of (1.7), (2.1), (2.14), (3.1) and (3.2), we find

$$K=4n(n-1)+h(h-\alpha).$$

We now prove from this that λ vanishes identically.

Since K is a constant because $\nabla_c K_{ba}-\nabla_b K_{ca}=0$, we have from this relation

(3.6) $$(\nabla_c h)(h-\alpha)+h\nabla_c(h-\alpha)=0.$$

On the other hand, we see from (3.4) that

(3.7) $$\nabla_b(\alpha-h)=\varepsilon u_b$$

for some function ε because the function $1-\lambda^2$ does not vanish.

If we differentiate (3.7) covariantly and substitute (1.9), then we obtain

$$\nabla_c\nabla_b(\alpha-h)=(\nabla_c\varepsilon)u_b+\varepsilon(\lambda g_{cb}+h_{ce}f_b{}^e),$$

which implies that

$$(\nabla_c\varepsilon)u_b-(\nabla_b\varepsilon)u_c+\varepsilon(h_{ce}f_b{}^e-h_{be}f_c{}^e)=0.$$

Transvecting this with u^b and taking account of (1.7), (2.1) and (2.14), we find

$$(1-\lambda^2)\nabla_c\varepsilon=(u^e\nabla_e\varepsilon)u_c-\lambda\alpha\varepsilon v_c.$$

Thus, it follows that

$$\frac{\lambda\alpha\varepsilon}{1-\lambda^2}(v_bu_c-v_cu_b)+\varepsilon(h_{ce}f_b{}^e-h_{be}f_c{}^e)=0.$$

Transvecting the above equation with f^{cb} gives $\varepsilon(h-\alpha)=0$. Hence (3.7) implies that $\nabla_b(\alpha-h)^2=0$ and consequently $\alpha-h$ is constant. Thus (3.6) reduces to $(\nabla_c h)(h-\alpha)=0$. Combining this fact and LEMMA 5, we see that λ vanishes identically and α is constant on M. Therefore (2.22) reduces to

(3.8) $$\frac{1}{2}\|\nabla_c u_b+\nabla_b u_c\|^2=h_{cb}h^{cb}-2n-\alpha h.$$

If we take account of (1.16), then (3.5) becomes

$$(h_c{}^a f_b{}^e - h_b{}^a f_c{}^e) h_{ea} = 2 f_{bc} + \alpha (h_{ce} f_b{}^e - h_{be} f_c{}^e).$$

Transvecting this with f^{cb} and using (1.7), (2.1) and (2.14), we obtain (2.26). Thus, due to LEMMA 7, we have (2.27).

Differentiating (2.27) covariantly and taking account of the fact that α is constant, we get

$$(\nabla_d h_{ce}) h_b{}^e + h_c{}^e (\nabla_d h_{be}) = \alpha \nabla_d h_{cb}. \tag{3.9}$$

Taking the skew-symmetric part with respect to d and c and using (1.15), we have

$$h_c{}^e (\nabla_d h_{be}) - h_d{}^e (\nabla_c h_{be}) = 0,$$

which implies that

$$h_c{}^e (\nabla_d h_{be}) - h_b{}^e (\nabla_c h_{de}) = 0. \tag{3.10}$$

Adding (3.9) to (3.10), we find

$$2 h_c{}^e (\nabla_d h_{be}) = \alpha \nabla_d h_{cb}. \tag{3.11}$$

Transvecting (3.11) with $h_a{}^c$ and using (2.27) and (3.11) itself, we get

$$\nabla_d h_{ba} = 0. \tag{3.12}$$

From (2.26) and (2.27) we easily see that the principal curvatures of the second fundamental tensor H are constant and the multiplicity of the principal curvature $(\alpha \pm \sqrt{\alpha^2+4})/2$ is odd. Developed above, we conclude that, by the completeness, M is a product of two spheres $S^p \times S^{2n-p}$, p being an odd number (cf. [10], [15]). This completes the proof.

According to THEOREM 8, we have

COROLLARY 9. *Let M be a complete and connected hypersurface with parallel Ricci tensor of an odd-dimensional sphere $S^{2n+1}(1)$ $(n>1)$. If the induced (f, g, u, v, λ)-structure is partially integrable, then M is a product of two spheres $S^p \times S^{2n-p}$, p being an odd number.*

REMARK. *C.-H. Chung and U-H. Ki [3] proved* COROLLARY *9 under the additional condition that the Sasakian structure vector of $S^{2n+1}(1)$ is tangent to the hypersurface.*

We next prove

THEOREM 10. *Let M be a complete and connected hypersurface of an odd-dimensional sphere $S^{2n+1}(1)$ $(n>1)$ such that the Ricci curvature S of M satisfies*

$$\nabla_X\nabla_Y S-\nabla_Y\nabla_X S+\nabla_{[X,Y]}S=0 \tag{3.13}$$

for any vector fields X and Y tangent to M. If the partially integrable (f, g, u, v, λ)-structure induced on M satisfies one of the following conditions:

(a) *M is compact,*

(b) *the mean curvature of M is constant,*

then M is of the same type as in THEOREM 8.

Proof. Transvecting (1.14) with v^d and using (2.14), we find

$$K_{dcba}v^d=v_a g_{cb}-v_b g_{ca}-u_a h_{cb}+u_b h_{ca}, \tag{3.14}$$

from which we have

$$K_{dcba}v^d v^b=v_c v_a+u_c u_a-(1-\lambda^2)g_{ca}, \tag{3.15}$$

transvecting v^b and taking account of (1.7) and (2.14).

From the Ricci identity for S, the local components of (3.13) is given by

$$K_{dcbe}K_a{}^e+K_{dcae}K_b{}^e=0. \tag{3.16}$$

Transvecting (3.16) with $v^d v^a$ and using (3.2), (3.14) and (3.15), we find

$$(v_e g_{cb}-v_b g_{ce}-u_e h_{cb}+u_b h_{ce})\{(2n-2)v^e+(\alpha-h)u^e\}$$
$$+K_{be}\{v_c v^e+u_c u^e-(1-\lambda^2)\delta_c^e\}=0.$$

Using (1.7), (2.1), (2.14), (3.1) and (3.2) we obtain

$$K_{cb}=(2n-2)g_{cb}+(h-\alpha)h_{cb} \tag{3.17}$$

because $1-\lambda^2$ is not zero, which implies that

$$K=4n(n-1)+h(h-\alpha). \tag{3.18}$$

Substituting (3.17) into (1.16), we get

$$h_{ce}h_b{}^e=\alpha h_{cb}+g_{cb}. \tag{3.19}$$

In the first place, if the mean curvature of M is constant, then, by LEMMA 3, we see that $\lambda=0$ and α is constant. Thus, as in the proof of THEOREM 8, it follows from (3.19) that

$$\nabla_d h_{cb}=0 \tag{3.20}$$

because of (1.15) and the fact that α=const. Hence (3.17) and (3.20) imply that the Ricci tensor of M is parallel.

In the next step, we suppose that M is compact. Since we have from (1.9)

$$\nabla_c u^c=2n\lambda,$$

it follows that λ vanishes identically on M because of LEMMA 1. Thus, α is constant on M by virtue of LEMMA 4. Therefore, we see from (3.19) that the second fundamental tensor of M is parallel and hence the Ricci tensor of M is parallel too.

According to COROLLARY 9, M is $S^p\times S^{2n-p}$, p being an odd number. This completes the proof.

Finally we prove

THEOREM 11. *Let M be a complete and connected hypersurface of $S^{2n+1}(1)$ $(n>1)$ with partially integrable (f, g, u, v, λ)-structure. If the mean curvature of M is constant, the second fundamental form H is positive semi-definite and satisfies Trace ${}^tHH\leqq 2n$ at every point on M, then M is $S^n\times S^n$ or $S^p\times S^{2n-p}$. In the latter case, p is odd.*

Proof. From LEMMA 3, we see that λ is zero and hence α is constant on M because of LEMMA 4. Since the second fundamental form is positive semidefinite, by the definition of α and h, it follows that $h\alpha$ is nonnegative. Thereby, from (2.22) we have

$$0\leqq\frac{1}{2}\|\nabla_c u_b+\nabla_b u_c\|^2\leqq h_{cb}h^{cb}-2n$$

which implies that

$$\nabla_c u_b+\nabla_b u_c=0, \tag{3.21}$$

$$h_{cb}h^{cb}=2n \tag{3.22}$$

because of $h_{cb}h^{cb}\leqq 2n$ at every point of M.

From (3.21) and LEMMA 7 we can prove that

$$h_{ce}h_b{}^e=g_{cb}+\alpha h_{cb} \tag{3.23}$$

because of (1.9), (2.1) and the fact that $\lambda=0$. This implies

$$h_{cb}h^{cb}=2n+\alpha h.$$

If we take account of (3.22), we obtain $\alpha h=0$ on M. Thus it follows that $\alpha=0$ or $h=0$ since α is constant.

First of all, if $h=0$, that is, M is minimal, then owing to Chern-do Carmo-Kobayashi's result [2], M is $S^n\times S^n$ because of (3.22).

Secondly if $\alpha=0$ on M, then (3.23) reduces to

$$h_{ce}h_b{}^e=g_{cb}. \tag{3.24}$$

From this and (1.15) we easily verify that $\nabla_d h_{cb}=0$.

On the other hand, from (3.24) we can see that the second fundamental tensor $h_c{}^a$ has two principal curvatures 1 and -1 and the multiplicities of these are odd. Thus M is a product of two odd-dimensional spheres. Hence THEOREM 11 is proved.

§4. Pseudo-Einstein hypersurfaces with partially integrable structure

Let M be a hypersurface of $S^{2n+1}(1)$. Then we have from (1.1) and (1.3)$\sim$(1.6)

$$\phi^2BX=B(-X+G(X,\ V)V)+\lambda G(X,\ V)C \tag{4.1}$$

for any vector field X tangent to M. We denote by ψ^2X the tangential part of ϕ^2BX.

If the Ricci tensor S of M is of the form

$$S(\psi^2X,\ \psi^2Y)=ag(\psi^2X,\ \psi^2Y)+bu(\psi^2X)u(\psi^2Y) \tag{4.2}$$

for any vector fields X and Y tangent to M, a and b being constants, then M is called a *pseudo-Einstein hypersurface* of $S^{2n+1}(1)$ (cf. [17]). In particular, if λ vanishes identically, then Yano and Kon called also the hypersurface is pseudo-Einstein.

Since $\psi^2X=-X+G(X,\ V)V$, (4.2) reduces to

$$\begin{aligned}K_{cb}=&(K_{ce}v^e)v_b+(K_{be}v^e)v_c-(K_{ed}v^ev^d)v_cv_b\\&+a\{g_{cb}-(1+\lambda^2)v_cv_b\}+bu_cu_b\end{aligned} \tag{4.3}$$

because of (1.3) and (1.7).

Transvecting (4.3) with v^b and using (1.7), we find

$$\lambda^2K_{ce}v^e=\lambda^2(K_{ed}v^ev^d)v_c+a\lambda^4v_c, \tag{4.4}$$

from which we have

$$\lambda^2(K_{ed}v^e v^d)=a\lambda^2(1-\lambda^2),$$

transvecting (4.4) with v^c.

Thus (4.4) becomes

$$\lambda^2 K_{ce}v^e=a\lambda^2 v_c.$$

Substituting this into (4.3), we get

$$\lambda^2 K_{cb}=\lambda^2(ag_{cb}+bu_c u_b). \tag{4.5}$$

In this section, we assume that M has partially integrable (f, g, u, v, λ)-structure. Then λ is constant on M because of LEMMA 1.

We now suppose that λ does not vanish. Then (4.5) implies

$$K_{cb}=ag_{cb}+bu_c u_b.$$

Substituting this into (1.16), we find

$$h_{ce}h_b^e=(2n-1-a)g_{cb}+hh_{cb}-bu_c u_b. \tag{4.6}$$

If we transvect (4.6) with u^b and take account of (2.1) and (2.14), we get

$$\{2n-2-a+h\alpha-\alpha^2-b(1-\lambda^2)\}u_c+(\alpha-h)v_c=0$$

and consequently $\alpha\neq h$ because $1-\lambda^2$ is not zero. Thus, by LEMMA 2, λ vanishes identically on M. Hence we have the pseuod-Einstein hypersurface in the sense of Yano and Kon.

According to THEOREM A and B we have

THEOREM 12. *Let M be a complete pseudo-Einstein hypersurface of $S^{2n+1}(1)$, $(n\geqq 3)$. If the induced structure on M is partially integrable, then M is congruent to some $M_0(2n, r)$ or to some $M(2n, m, (m-1)/(n-m))$ or to $M(2n, 1/(n-1))$.*

THEOREM 13. *Let M be a complete pseudo-Einstein minimal hypersurface of $S^{2n+1}(1)$, $(n\geqq 3)$. If the induced structure on M is partially integrable, then M is congruent to $M(2n, 2n-1)$ or to $M(2n, (n+1)/2, 1)$. In the latter case, n is odd.*

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