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HOMOMORPHISMS ON BANACH ALGEBRAS

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1. Introduction

The automatic continuity problem for Banach algebras is usually formulated in terms of two classes of mappings. First, if θ is a homomorphism from a Banach algebra A into a Banach algebra B, then what conditions on A and B ensure that θ is continuous? Sencondly, if D is a derivation in A, then under what conditions on A will D be continuous? By a derivation in A we mean a linear mapping in A satisfying the derivative identity,

$$D(ab)=aDb+(Da)b \qquad (a, b\in A) \quad .$$

In [9] Singer and Wermer showed that the range of a continuous derivation on a commutative Banach algebra is contained in the radical. They conjectured that the assumption of continuity is unnecessary.

In [3] Cusack also showed that if D is a derivation on a commutative Banach algebra A such that the separating space of D is nilpotent, then $D(A)$ is contained in the radical of A.

In this paper we establish the algebraic conditions on a Banach algebra B so that the homomorphism $\theta : A\to B$ is necessarily continuous, and we also generalize the results of Cusack.

2. Separating spaces and separating ideals

In this section we present preliminary facts which will be used in the later sections.

Let X and Y be Banach spaces, and let S be a linear mapping from X into Y. Then the separating space of S is the set

$$\mathcal{G}(S)=\{y\in Y\colon \text{ there exists } x_n\to 0 \text{ in } X \text{ with } Sx_n\to y \text{ in } Y\}$$

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The proofs of the following lemmas 2.1 to 2.4 are given in [8].

LEMMA 2.1 *$\mathcal{G}(S)$ is a closed linear subspace of Y, and S is continuous if and only if $\mathcal{G}(S)=\{0\}$.*

LEMMA 2.2. *Let X, Y and Z be Banach spaces, let S be a linear mapping from X into Y, and let R be a continuous linear mapping from Y into Z. Then*

(1) RS is continuous if and only if $R\mathcal{G}(S)=\{0\}$

(2) $(R\mathcal{G}(S))^{-}=\mathcal{G}(RS)$

LEMMA 2.3. *Let S be a linear mapping from a Banach space X into a Banach space Y, and X_0 and Y_0 be closed linear subspaces of X and Y respectively, such that SX_0 is contained in Y_0. If $S_0 : X/X_0 \rightarrow Y/Y_0$ is defined by*

$$S_0(x+x_0)=Sx+Y_0 \qquad (x\in X),$$

then S_0 is continuous if and only if $\mathcal{G}(S)$ is contained in Y_0.

LEMMA 2.4. *Let X and Y be Banach spaces, let S be a linear mapping from X into Y, and let $\{T_n\}$ and $\{R_n\}$ be sequences of bounded linear operators on X and Y respectively, such that ST_n-R_nS is continuous for all n. Then there is a natural number N such that*

$$(R_1\cdots R_n\mathcal{G}(S))^{-}=(R_1\cdots R_N\mathcal{G}(S))^{-} \quad (n\geq N).$$

Let B be a Banach algebra. Then a subset J of B is called a separating ideal of B if is a closed ideal of B with the property that, for every sequence $\{b_n\}$ in B, there exists a natural number N such that

$$(Jb_n\cdots b_1)^{-}=(Jb_N\cdots b_1)^{-} \qquad (n\geq N).$$

Note that any finite-demensional ideal of a Banach algebra is a separating ideal. The following lemma shows that the separting space of an epimorphism from a Banach algebra A onto a Banach algerbra B, or of a derivation on B, is a separating ideal of B, which is due to Jewell and Sinclair [5].

LEMMA 2.5. *Let S be a linear mapping from a Banach space X into a Banach algebra B. Suppose that there exist continuous linear operators T_b and U_b on X, for all b in B, such that the maps*

$$x\rightarrow ST_bx\rightarrow(Sx)b \text{ and } x\rightarrow SU_bx-b(Sx)$$

from X into B are continuous. Then the separating space $\mathcal{G}(S)$ of S is a separating ideal of B.

Proof. By Lemma 2.1, $\mathcal{G}(S)$ is a closed linear subspace of B. Let a be any element of $\mathcal{G}(S)$, and let $\{x_n\}$ be a sequence in X such that $x_n \to 0$ and $Sx_n \to a$.
Then, for all b in B,

$$ST_b x_n - (Sx_n)b \to 0 \text{ and } SU_b x_n - b(Sx_n) \to 0.$$

Thus, $ab = \lim ST_b x_n$ and $ba = \lim SU_b x_n$ and therefore, ab and ba are in $\mathcal{G}(S)$. This proves that $\mathcal{G}(S)$ is an ideal.

Now let $\{b_n\}$ be any sequence in B, and let

$$R_n a = ab_n \text{ and } T_n = T_{bn} \quad (a \in B, \ n = 1, 2, \cdots).$$

By Lemma 2.4, there exists a natural number N such that

$$(R_1 \cdots R_n \ \mathcal{G}(S))^- = (R_1 \cdots R_N \ \mathcal{G}(S))^- \quad (n \geq N).$$

Since $R_1 \cdots R_n \mathcal{G}(S) = \mathcal{G}(S) b_n \cdots b_1$, this completes the proof.

COROLLARY 2.6. *Let θ be an epimorphism from a Banach algebra A onto a Banach algebra B. Then the separating space $\mathcal{G}(\theta)$ of θ is a separating ideal of B.*

Proof. For b in B, let

$$T_b x = xa \text{ and } U_b x = ax \quad (x \in A),$$

where a is some element of A such that $\theta(a) = b$. Now apply Lemma 2.5, with $X = A$ and $\theta = S$.

COROLLARY 2.7. *Let D be a derivation on a Banach algebra A. Then the separating space $\mathcal{G}(D)$ of D is a separating ideal of A.*

Proof. In this case, we have $X = A = B$, $D = S$, and for all b and x in A, $T_b x = xb$ and $U_b x = bx$. Thus, by defining condition for a derivation,

$$D(T_b x) - (Dx)b = T_{D(b)} x \text{ and}$$
$$D(U_b x) - bDx = U_{D(b)} x \quad (x, b \in A).$$

Since $T_{D(b)}$ and $U_{D(b)}$ are continuous, the conditions of Lemma 2.5 are satisfied. Therefore $\mathcal{G}(D)$ is a separating ideal of A.

3. Homomorphisms on Banach algebras.

The following lemma is well known. The proof is given in [2].

LEMMA 3.1. *Let θ be an epimorphism from a Banach algebra A onto a Banach algebra B. Then the separating space of θ is contained in the radical of B.*

The following lemma is due to Grabiner [4]. The proof is also given in [1, Theorem 46.3]

LEMMA 3.2. *A nil Banach algebra is nilpotent.*

THEOREM 3.3. *Let θ be an epimorphism from a Banach algebra A onto a Banach algebra B. If the radical R of B satisfies $\bigcap_{n\geq 1} R^n=\{0\}$, then the separating space of θ is nilpotent.*

Proof. Let $\mathcal{G}$ be the separating space of θ. Let x be an element of $\mathcal{G}$. Then, by the Lemma 3.1, x belongs to R, the radical of B. By the Corollary 2.6, $\mathcal{G}$ is also a separating ideal. So there exists a natural number N such that

$$(\mathcal{G}x^N)^-=(\mathcal{G}x^n)^- \text{ for all } n\geq N.$$

Since $\mathcal{G}x^{n+1}\subseteq\mathcal{G}x^n$ for all n,

$$(\mathcal{G}x^N)^-=\bigcap_{n\geq 1}(\mathcal{G}x^n)^-\subseteq\bigcap_{n\geq 1}\mathrm{R}^n=\{0\}.$$

This implies that $x^{N+1}=0$ and so $\mathcal{G}$ is nil. Since $\mathcal{G}$ is closed, $\mathcal{G}$ is nil potent by Lemma 3.2.

COROLLARY 3.4. *Let θ be an epimorphism from a Banach algebra A onto a Banach algebra B. If the radical R of B satisfies $\bigcap_{n\geq 1} R^n=\{0\}$ and is an integral domain, then θ is continuous.*

Proof. By Theorem 3.3, the separating space $\mathcal{G}$ of θ is nilpotent. Since R is an integral domain and $\mathcal{G}\subseteq R$ by Lemma 3.1, $\mathcal{G}=\{0\}$ and hence θ is continuous.

Recall that semi-prime Banach algebra is a Banach algebra that has no non-zero nil ideals [1, Corollary 46.5]

COROLLARY 3.5. *Let θ be an epimorphism from a Banach algebra A onto a semi-prime Banach algebra B. If the radical R of B satisfies*

$\bigcap_{n\geq 1} R^n = \{0\}$, *then θ is continuous.*

Proof. By Theorem 3.3, the separating space $\mathcal{G}$ of θ is nilpotent. Since B is semi-prime, we have $\mathcal{G} = \{0\}$ and so θ is continuous.

4. Derivations on Banach algebras

In [3] Cusack has shown that if D is a derivation on a commutative Banach algebra A such that the separating space $\mathcal{G}(D)$ of D is nilpotent, then $D(A)$ is contained in the radical R of A. We generalize this result.

The following lemma is well known. The proof is given in [6].

LEMMA 4.1. *Let J be a separating ideal of a semi-simple Banach algebra. Then J is finite-dimensional.*

The following lemma is an immediate consequence of the Wedderburn structure theorem of finite-dimensional semi-simple algebras, which follows from Jacobson's density theorem [1].

LEMMA 4.2 *Let J be a finite-demensional semi-simple subalgebra of A, where A is an algebra. Then J has an identity element e. If J is an ideal, then $J = Ae$ and e commutes with every element of A.*

We need another lemma. The proof is given in [3].

LEMMA 4.3. *Let D be a derivation on a Banach algebra A, and let L be the prime radical of A. Then $D(L)$ is contained in L.*

Now we have the main theorem.

THEOREM 4.4. *Let A be a commutative Banach algebra with radical R. Let D be a derivation in A with a separating space $\mathcal{G}$. If there exists an ideal I of A such that $I \subseteq R$, $D(I) \subseteq I$ and $\mathcal{G} \cap R \subseteq I$, then $D(A) \subseteq R$.*

Proof. We use the argument of Cusack [3]. Let Q be the natural homomorphism of A onto $A/\mathcal{G} \cap R$. Suppose that $\mathcal{G} \neq \mathcal{G} \cap R$. Then by Lemma 4.1, $Q\mathcal{G}$ is nonzero finite dimensional semi-simple algebra. By Lemma 4.2, there exists an element $e \in \mathcal{G}$ such that Qe is an identity element for $Q\mathcal{G}$. Let $e_1, \cdots, e_n$ be elements of $\mathcal{G}$ such that $Qe_1, \cdots, Qe_n$ is a basis for $Q\mathcal{G}$. Then there exist continuous linear functionals $f_1, \cdots, f_n$ on $Q\mathcal{G}$ such that

$$Qa = \sum_{i=1}^{n} f_i(Qa) Qe_i \quad (a \in \mathcal{G}).$$

In particular,

$$ea - \sum_{i=1}^{n} f_i(Qea) e_i \in \mathcal{G} \cap Q \subseteq I \quad (a \in A).$$

Since $e \in \mathcal{G}$, there exists a sequence $\{a_k\}$ in A such that

$$a_k \to 0 \text{ and } Da_k \to e \text{ as } k \to \infty.$$

This implies that

$$D(ea_k) - \sum_{i=1}^{n} f_i(Qea_k) D(e_i) \in D(I)$$

since $D(I) \subseteq I$. For $i = 1, \cdots, n$, $f_i(Qea_k) \to 0$. Since $D(ea_k) = eDa_k + (De) a_k \to e^2$, it follows that e^2 is in closure $\bar{I}$ of I. Thus $e^2 \in R$. Therefore $0 = Qe^2 = (Qe)^2 = Qe$, which is a contradiction. Hence $\mathcal{G} = \mathcal{G} \cap R$.

Now let $\bar{Q}$ be the natural homomorphism of A onto $A/\bar{I}$. Then $\bar{Q}D$ is continuous since $\mathcal{G} \subseteq I$. $D(I) \subseteq I$ implies that $\bar{Q}D(I) = \{0\}$. Hence $\bar{Q}D(\bar{I}) = \{0\}$. Thus $D(\bar{I}) \subseteq \bar{I}$. So we can define the map D_0 on $A/\bar{I}$ by $D_0(a + \bar{I}) = Da + \bar{I}$. Then D_0 is a continuous derivation by Lemma 2.3. Singer and Wermer's Theorem [9] implies that $D_0(A/\bar{I}) \subseteq R/\bar{I}$. So $D(A) \subseteq R$. This completes the proof.

We get the Cusack's result as a corollary.

COROLLARY 4.5. *If D is a derivation on a commutative Banach algebra A such that the separating space $\mathcal{G}$ of D is nilpotent, then $D(A) \subseteq R$.*

Proof. Let L be the prime radical of A. By Lemma 4.3, $D(L) \subseteq L$. Clearly $L \subseteq R$. Since $\mathcal{G}$ is nilpotent, $\mathcal{G} \subseteq L$. Thus the conditions of Theorem 4.4 are satisfied and hence we have $D(A) \subseteq R$.

The following corollary has been observed by Khosravi [7].

COROLLARY 4.6. *Let A be a commutative Banach algebra with radical R and D be a derivation on A with a separating space $\mathcal{G}$. Let $I = \{x \in R$: for every $n \geq 1$, $D^n x \in R\}$. If $\mathcal{G} \cap R \subseteq I$, then $D(A) \subseteq R$.*

Proof. Clearly $I \subseteq R$ *and* $D(I) \subseteq I$. By the hypothesis, $\mathcal{G} \cap R \subseteq I$.

Hence the result holds by Theorem 4.4.

COROLLARY 4.7. *If D is a derivation in a commutative Banach algebra A such that $D(\mathcal{G}\cap R)\subseteq G\cap R$, then $D(A)\subseteq R$.*

Proof. Putting $I=\mathcal{G}\cap R$, it satisfies the conditions of Theorem 4.4.

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