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ON CLASSES OF MORPHISMS CLOSED UNDER LIMITS

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1. Introduction

When we say below that a full subcategory $\mathscr{C}$ of a category $\mathscr{B}$ is *closed under limits* we mean that, whenever a functor $F:\mathscr{K}\to\mathscr{C}$ is such that the composite $\mathscr{K}\to\mathscr{C}\to\mathscr{B}$ admits a limit $(\alpha_K : A\to FK)$ in $\mathscr{B}$, then the object $A=\lim F$ itself lies in $\mathscr{C}$; so that $\mathscr{C}$ is necessarily a ***replete*** full subcategory of $\mathscr{B}$.

It should cause no confusion if we use the phrase "closed under limits" in another but related way. We say that a class $\mathscr{M}$ of morphisms in a category $\mathscr{A}$ is *closed under limits* if, whenever $F, G:\mathscr{K}\to\mathscr{A}$ are functors that admit limits, and whenever $\eta: F\to G$ is a natural transformation each of whose components $\eta_K: FK\to GK$ lies in $\mathscr{M}$, then the induced morphism $\lim\eta:\lim F\to\lim G$ also lies in $\mathscr{M}$. If we also use $\mathscr{M}$ to denote that full subcategory of $\mathscr{A}^2$ (the category of morphisms in $\mathscr{A}$ and commutative squares) whose objects are the elements of $\mathscr{M}$, then to say that $\mathscr{M}$ is closed under limits in $\mathscr{A}$ is to say that the full subcategory $\mathscr{M}$ is closed in $\mathscr{A}^2$, in the sense of the preceding paragraph, not in general under *all* limits, but under all *pointwise* limits-which are the only ones of interest in a functor category. This is of course a *fortiori* the case when the subcategory $\mathscr{M}$ is *replete* and *reflective* in $\mathscr{A}^2$.

More generally, we may speak similarly of the class $\mathscr{M}$ as being closed, not under all limits, but under some class of limits, such as small ones, or finite ones, and so on. This leads us to warn against a danger of misunderstanding. Sometimes we wish to say that every pullback in $\mathscr{A}$ of a morphism in $\mathscr{M}$ itself lies in $\mathscr{M}$; we can express this property by saying that $\mathscr{M}$ is *stable under pullbacks*; but we cannot, like some auth-

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ors, express it by saying that "$\mathcal{M}$ is closed under pullbacks"–since, for us, this last has quite a different meaning.

The aim of the present article is to observe that any class $\mathcal{M}$ of morphisms which *contains the identities and is closed under limits* necessarily enjoys a large number of other closure properties. This observation seems to be new; for many authors, even recent ones, and even those who have explicitly noted the closedness under limits of their class $\mathcal{M}$, have provided independent proofs of a variety of these other closure properties, which are in fact consequences. See for example Kelly [4], Section 3 of Ringel [7], Freyd–Kelly [2], Bousfield [1], MacDonald–Tholen [6], and Tholen [8], among others. The last two authors have in fact shown some of these further closure properties to follow from the stronger hypothesis that $\mathcal{M}$, besides containing the identities, is *replete and reflective* in $\mathcal{A}^2$. It should be pointed out that not even this stronger hypothesis implies that $\mathcal{M}$ is *closed under composition.*

On the case where $\mathcal{M}$ is reflective in $\mathcal{A}^2$, and on the case where $\mathcal{M}$ is closed under composition, we have not very much to add to the results in the articles of MacDonald and Tholen above. Not to say something on these, however, would leave the reader without a clear view of the situation. Accordingly we take the opportunity, in the final two sections below, to include some comments on these cases which go a little beyond those of these authors, as well as making explicit some ideas that are only implicit in their articles, or are divided between their two articles and not wholly contained in either.

2. Some consequences of closedness under limits and identities

We give some names to certain closure properties of a class $\mathcal{M}$ of morphisms of $\mathcal{A}$.

M 1. $\mathcal{M}$ contains all identities.

M 2. $\mathcal{M}$ is closed under limits.

M 3. Any retract in $\mathcal{A}^2$ of an $\mathcal{M}$ is an $\mathcal{M}$.

M 4. Any isomorph in $\mathcal{A}^2$ of an $\mathcal{M}$ is an $\mathcal{M}$; equivalently, $\mathcal{M}$ is replete in $\mathcal{A}^2$, or $umv \in \mathcal{M}$ whenever $m \in \mathcal{M}$ and u, v are invertible.

M 5. If $fr \in \mathcal{M}$ and r is a retraction, then $f \in \mathcal{M}$.

M 6. $\mathcal{M}$ contains all the isomorphisms.

M 7. If $fmg \in \mathcal{M}$, $fm \in \mathcal{M}$, and $m \in \mathcal{M}$, then $mg \in \mathcal{M}$.

M 8. If $fg \in \mathcal{M}$ and $f \in \mathcal{M}$, then $g \in \mathcal{M}$.

M 9. If $m \in \mathcal{M}$, $g \in \mathcal{M}$, and m is a coretraction, then $mg \in \mathcal{M}$.

M10. Let $\mathcal{K}$ have a terminal object 1, and write $t_K : K \to 1$ for the unique map. If $F : \mathcal{K} \to \mathcal{A}$ has a limit $(\alpha_K : A \to FK)$, and if $Ft_K \in \mathcal{M}$ for each K, then $\alpha_K \in \mathcal{M}$ for each K.

M11. If a family $(m_i : B_i \to C)_{i \in I}$ with each $m_i \in \mathcal{M}$ admits a fibred product

$$\begin{tikzcd} & B_i \arrow[dr, "m_i"] & \\ A \arrow[ur, "g_i"] \arrow[rr, "h"'] & & C, \end{tikzcd} \tag{2.1}$$

then $h \in \mathcal{M}$ and each $g_i \in \mathcal{M}$.

M12. If, in a fibred product (2.1), we have $m_i \in \mathcal{M}$ for every i except for one value $i = 0 \in I$, then $g_0 \in \mathcal{M}$.

M13. Every pullback of an $\mathcal{M}$ is an $\mathcal{M}$.

M14. If $fg \in \mathcal{M}$ and f is monomorphic, then $g \in \mathcal{M}$.

We begin with some simple observations.

LEMMA 2.1. M3 *contains* M4 *and* M5 *as special cases, while either* M4 *or* M5 *implies* M6 *in the presence of* M1.

Proof. The only non-trivial observation needed is that, in the circumstances of M5, if $ri = 1$, the equation

$$\begin{tikzcd} \cdot \arrow[r, "i"] \arrow[d, "f"'] & \cdot \arrow[r, "r"] \arrow[d, "fr"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "1"'] & \cdot \arrow[r, "1"'] & \cdot \end{tikzcd} \quad = \quad \begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "f"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

exhibits f as a retract in $\mathcal{A}^2$ of fr.

LEMMA 2.2. M7 *contains* M8 *and* M9 *as special cases in the presence of* M1.

Proof. For M8, take $m = 1$ in M7. For M9, so choose f in M7 that $fm = 1$.

LEMMA 2.3. M11 *is a special case of* M10.

LEMMA 2.4. M13 *is the special case* $I=2$ *of* M12, *while* M14 *is a special case of* M13.

Proof. For the latter statement, observe that

$$\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "1"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "fg"'] & \cdot \end{tikzcd}$$

is a pullback if f is monomorphic.

THEOREM 2.5. M2, *even stated for finite limits only, implies* M3 *and* M7; *while* M1 *and* M2 *together imply* M10 *and* M12 (*with* M2 *being required only for small limits, or only for finite ones, if the category* $\mathscr{K}$ *or the set* I *is small or finite*). *Hence* M1 *and* M2 *together imply* M3–M14.

Proof. In the case of M3 we are contemplating the situation

$$\begin{tikzcd} \cdot \arrow[r, "i_0"] \arrow[d, "f"'] & \cdot \arrow[r, "r_0"] \arrow[d, "m"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "i_1"'] & \cdot \arrow[r, "r_1"'] & \cdot \end{tikzcd}$$

where $r_0 i_0=1$, $r_1 i_1=1$, and $m \in \mathscr{M}$. Writing i for the map (i_0, i_1) of $\mathscr{A}^2$ and so on, we have in $\mathscr{A}^2$ the pointwise equalizer

$$f \xrightarrow{i} m \underset{1}{\overset{ir}{\rightrightarrows}} m,$$

so that $f \in \mathscr{M}$ by M2. For M7 we consider a diagram

$$\begin{tikzcd} & A \arrow[rr, "g"] \arrow[dd, "fmg"] & & B \arrow[dd, "fm"] \\ A \arrow[ur, "1"] \arrow[rr, "g"] \arrow[dd, "mg"'] & & B \arrow[ur, "1"] \arrow[dd, "m"] & \\ & D \arrow[rr, "1"] & & D \\ C \arrow[ur, "f"] \arrow[rr, "1"'] & & C \arrow[ur, "f"'] & \end{tikzcd}$$

since the top and bottom squares are pullbacks, M2 gives $mg \in \mathscr{M}$ from the hypotheses of M7. Turning to M10, we consider the functor $\Delta(F1)$: $\mathscr{K} \to \mathscr{A}$ constant at $F1$; since $\mathscr{K}$ is connected, its limit is $F1$, the limit-cone having generators $1 : F1 \to F1$. The Ft_K are the components of a natural transformation $\eta : F \to \Delta(F1)$, with each η_K in $\mathscr{M}$ by hypothesis. Clearly $\lim \eta : A \to F1$ is α_1; so that $\alpha_1 \in \mathscr{M}$ by M2. Since $\alpha_1 = Ft_K \cdot \alpha_K$, it follows from M8 that $\alpha_K \in \mathscr{M}$. Passing finally to M12, we consider

$$\begin{tikzcd}
& B_i \arrow[dr, "m_i"] \arrow[dd, "k_i"] & \\
A \arrow[ur, "g_i"] \arrow[rr, "h"] \arrow[dd, "g_0"'] & & C \arrow[dd, "1"] \\
& D_i \arrow[dr, "t_i"] & \\
B_0 \arrow[ur, "s_i"] \arrow[rr, "m_0"'] & & C
\end{tikzcd}$$

where, for $i \neq 0$, $D_i = C$, $k_i = m_i$, $t_i = 1$, and $s_i = m_0$; while $D_0 = B_0$, $k_0 = 1$, $t_0 = m_0$, and $s_0 = 1$. Since the bottom like the top is a fibred product, M1 and M2 give $g_0 \in \mathscr{M}$.

3. Closure under limits when $\mathscr{A}$ admits pullbacks

It is instructive to observe that, when $\mathscr{A}$ admits pullbacks and $\mathscr{M}$ contains the identities, there is an alternative way of expressing the closedness of $\mathscr{M}$ under limits, in terms of the categories $\mathscr{A}/A$ of objects over A for $A \in \mathscr{A}$.

We consider limits in $\mathscr{A}/A$. For any $\mathscr{K}$ we write $\Delta A : \mathscr{K} \to \mathscr{A}$ for the functor constant at A, and we write $\mathscr{K}^+$ for the category obtained from $\mathscr{K}$ by adding a new terminal object 1. To give a functor $\Phi : \mathscr{K} \to \mathscr{A}/A$ is equally to give a functor $H : \mathscr{K} \to \mathscr{A}$ and a natural transformation $\lambda : H \to \Delta A$, which in turn is equally to give a functor $\Phi^+ : \mathscr{K}^+ \to \mathscr{A}$ with $\Phi^+(1) = A$. To give a (projective) cone over Φ with vertex $n : B \to A$ is to give a cone $\alpha : \Delta B \to H$ in $\mathscr{A}$ satisfying

$$\begin{tikzcd}
\Delta B \arrow[r, "\alpha"] \arrow[dr, "\Delta n"'] & H \arrow[d, "\lambda"] \\
& \Delta A,
\end{tikzcd} \qquad (3.1)$$

which is equally to give a cone $(\alpha, n) : \Delta B \to \Phi^+$ in $\mathscr{A}$. It is clear that:

LEMMA 3.1. *α exhibits n as the limit of $\Phi = (H, \lambda) : \mathscr{K} \to \mathscr{A}/A$ if and only if (α, n) exhibits B as the limit of $\Phi^+ : \mathscr{K}^+ \to \mathscr{A}$.*

Suppose now that $\mathscr{A}$ admits pullbacks, and consider a natural transformation $\eta : F \to G : \mathscr{K} \to \mathscr{A}$, where F and G admit limits $\rho : \Delta B \to F$ and $\sigma : \Delta A \to G$, while η induces $n = \lim\, \eta : B \to A$. Form in the functor category $[\mathscr{K}, \mathscr{A}]$ the pullback H of η and σ, and let α be the unique map rendering commutative

$$\begin{tikzcd}
\Delta B \arrow[rr, "\rho"] \arrow[dr, "\alpha"] \arrow[dd, "\Delta n"'] & & F \arrow[dd, "\eta"] \\
& H \arrow[ur, "\mu"'] \arrow[dl, "\lambda"] & \\
\Delta A \arrow[rr, "\sigma"'] & & G
\end{tikzcd} \tag{3.2}$$

The reader will easily verify that:

LEMMA 3.2. *α expresses n as the limit of $\Phi = (H, \lambda) : \mathscr{K} \to \mathscr{A}/A$.*

The result we desire is:

PROPOSITION 3.3. *Let $\mathscr{A}$ admit pullbacks and let the class $\mathscr{M}$ of morphisms contain the identities. Then $\mathscr{M}$ is closed under limits if and only if $\mathscr{M}$ is stable under pullbacks and, for each $A \in \mathscr{A}$, the full subcategory $\mathscr{M}/A$ of $\mathscr{A}/A$, determined by those $m : C \to A$ with $m \in \mathscr{M}$, is closed under limits in $\mathscr{A}/A$.*

Proof. If $\mathscr{M}$ is closed under limits it is stable under pullbacks by the M13 part of Theorem 2.5; while if $\Phi = (H, \lambda) : \mathscr{K} \to \mathscr{A}/A$ takes it values in $\mathscr{M}/A$, and has limit n as in (3.1), we have $\lambda \in \mathscr{M}$ (componentwise); whence we deduce $n \in \mathscr{M}$ by Lemma 3.1 and the M10 part of Theorem 2.5 applied to $\Phi^+ : \mathscr{K}^+ \to \mathscr{A}$. For the converse, if $\eta \in \mathscr{M}$ (componentwise) in (3.2), we have $\lambda \in \mathscr{M}$ by the stability hypothesis and then $n \in \mathscr{M}$ by the other hypothesis and Lemma 3.2.

4. Examples of families of morphisms closed under limits

PROPOSITION 4.1. *The monomorphisms are closed under limits and contain the identities; they are also closed under composition.*

Recall from [4] that $j: A \to B$ is called a ***regular monomorphism*** if it is the joint equalizer of some family (not necessarily small) of pairs x_i, $y_i: B \to C_i$. It comes to the same thing to say that j is the joint equalizer of the family of *all* pairs $x, y: B \to C_{xy}$ satisfying $xj = yj$. By an *equalizer* we mean a morphism j that is the equalizer of a single pair $x, y: B \to C$; thus every equalizer (and so, in particular, every coretraction) is a regular monomorphism. On the other hand every regular monomorphism j is an equalizer if $\mathscr{A}$ admits pushouts; for then j is the equalizer of the pair x, y arising from the pushout of j by itself. As is well known, regular monomorphisms are not in general closed under composition.

PROPOSITION 4.2. *The regular monomorphisms contain the identities and are closed under limits.*

Proof. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha_K"] \arrow[d, "n"'] & FK \arrow[d, "\eta_K"] \\ B \arrow[r, "\beta_K"'] & GK \end{tikzcd}$$

where the top and the bottom are limits of F and of G, and the components η_K of $\eta: F \to G$ are regular monomorphisms. Suppose that η_K is the joint equalizer of the family

$$(x(K,i), y(K,i): GK \to H(K,i))_{i \in I_K};$$

then n is in fact the joint equalizer of the family

$$\Big(B \xrightarrow[\beta_K]{} GK \underset{y(K,i)}{\overset{x(K,i)}{\rightrightarrows}} H(K,i)\Big)_{K \in \mathscr{K},\, i \in I_K},$$

as the reader will easily verify.

Recall from [2] the notion of a ***prefactorization system***. For morphisms p and j, we write $p \downarrow j$ if, whenever we have a commutative square $ju = vp$, there is a unique "diagonal" w satisfying $wp = u$ and $jw = v$. For any class $\mathscr{N}$ of morphisms we write $\mathscr{N}^{\downarrow}$ for $\{j \mid p \downarrow j$ for all $p \in \mathscr{N}\}$ and $\mathscr{N}^{\uparrow}$ for $\{p \mid p \downarrow j$ for all $j \in \mathscr{N}\}$. A ***prefactorization system*** is a pair $\mathscr{E}, \mathscr{M}$ of classes of morphisms such that $\mathscr{E} = \mathscr{M}^{\uparrow}$ and $\mathscr{M} = \mathscr{E}^{\downarrow}$; and any class $\mathcal{N}$ gives rise to such a prefactorization system on setting $\mathcal{M}=\mathcal{N}^{\downarrow}$ and $\mathcal{E}=\mathcal{M}^{\uparrow}=\mathcal{N}^{\downarrow\uparrow}$. Many closure properties of a class defined by $\mathcal{M}=\mathcal{N}^{\downarrow}$ were given in [2], but it was Bousfield [1] who first pointed out that such an $\mathcal{M}$ is closed under limits.

There is a generalization of this notion due to Tholen; see for instance [8]. Following Tholen, define a *factorization* to be a pair (p, k), where p is a family $(p_i : X_i \to Y)_{i\in I(p)}$ of morphisms and $k : Y\to Z$ is a morphism; the idea is that (p, k) can be seen as a "*factorization*" of the family $(kp_i : X_i \to Z)$. For a morphism $m : A\to B$ write $(p, k) \downarrow m$ if, for all u_i, v rendering commutative the exterior of

$$\begin{tikzcd}
X_i \arrow[r, "p_i"] \arrow[d, "u_i"'] & Y \arrow[r, "k"] \arrow[dl, dashed, "w"] & Z \arrow[d, "v"] \\
A \arrow[rr, "m"'] & & B,
\end{tikzcd} \tag{4.1}$$

there is a unique w as shown rendering the square and the triangles commutative. If $\mathcal{F}$ is any class of factorizations, we set $\mathcal{F}^{\downarrow} = \{m \mid (p, k) \downarrow m$ for all $(p, k) \in \mathcal{F}\}$. If we identify a factorization $(p, 1_Y)$ with the family $p=(p_i : X_i \to Y)$, and identify a one-object family with a morphism $p : X\to Y$, it is clear that our definition of $\mathcal{F}^{\downarrow}$ contains as a special case that of $\mathcal{N}^{\downarrow}$ above.

The reader will have no difficulty in verifying the following result of Tholen [8] (see his Proposition 1.1 on page 66 and Proposition 2.1 on page 68):

PROPOSITION 4.3. *For any class $\mathcal{F}$ of factorizations, the class $\mathcal{M}=\mathcal{F}^{\downarrow}$ of morphisms contains the identities and is closed under limits. If every $(p, k) \in \mathcal{F}$ has k an identity, $\mathcal{M}$ is also closed under composition. This last is so in particular if $\mathcal{F}$ reduces as above to a class $\mathcal{N}$ of morphisms.*

Recall from [4] that the class of *strong monomorphisms* is the intersection of $\mathcal{N}^{\downarrow}$ with the monomorphisms, where $\mathcal{N}$ is the class of epimorphisms; it is accordingly closed under limits and composition, as well as containing the identities. (In this case, it is shown in [2] that $\mathcal{M}^{\downarrow}$ is already contained in the monomorphisms if $\mathcal{A}$ admits pullbacks or binary coproducts; it is also clearly so if $\mathscr{A}$ admits coequalizers).

If we take for $\mathfrak{F}$ the class of all $(p, 1_Y)$ where $p=(p_i : X_i \to Y)$ is a jointly epimorphic family, we call the intersection of $\mathfrak{F}^{\downarrow}$ with the monomorphisms the class of *familially strong* monomorphisms; clearly it too has the closure properties above. It is of course contained in the class of strong monomorphisms: we examine in the forthcoming [3] cases where these two classes coincide. For the moment we merely observe that:

PROPOSITION 4.4. *Every regular monomorphism is familially strong.*

Proof. In (4.1) let $p=(p_i)$ be jointly epimorphic, let $k=1_Y$, and let m be a regular monomorphism. Then $xm=ym$ gives $xvp_i=yvp_i$ for each i, whence $xv=yv$. So $v=mw$ for some unique w, and moreover $wp_i=u_i$ since m is monomorphic.

We give no concrete example here of an $\mathfrak{F}^{\downarrow}$ where the $(n,k)\in\mathfrak{F}$ do not all have k invertible; but Proposition 5.7 below shows that any $\mathscr{M}$ containing the identities and replete and reflective in $\mathscr{A}^2$ is $\mathfrak{F}^{\downarrow}$ for a class $\mathfrak{F}$ of factorizations–indeed of factorizations of single morphisms, rather than of families. In any reasonable $\mathscr{A}$ the regular monomorphisms form such a class $\mathscr{M}$ (see Example 6.3 below), but are not usually closed under composition: so that here the $(n,k)\in\mathfrak{F}$ do not have k invertible.

5. Reflexions of morphisms into $\mathscr{M}$

PROPOSITION 5.1. *Let $\mathscr{M}$ be a class of morphisms in $\mathscr{A}$ that contains the identities, identified with a full subcategory of $\mathscr{A}^2$. If an object f : $A \to B$ of $\mathscr{A}^2$ admits a reflexion*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[d, "x"] \\ C \arrow[r, "m"'] & D \end{tikzcd} \tag{5.1}$$

into $\mathscr{M}$, then x is invertible.

Proof Since $1_B\in\mathscr{M}$ and (5.1) is the reflexion, there are z, y such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[d, "x"] \\ C \arrow[r, "m"] \arrow[d, "z"'] & D \arrow[d, "y"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "1"] \\ B \arrow[r, "1"'] & B \end{tikzcd} . \tag{5.2}$$

Composing with $(x, x) : 1_B \to 1_D$ and using (5.1) gives

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[d, "x"] \\ C \arrow[r, "m"] \arrow[d, "z"'] & D \arrow[d, "y"] \\ B \arrow[r, "1"] \arrow[d, "x"'] & B \arrow[d, "x"] \\ D \arrow[r, "1"'] & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f"] \arrow[dd, "f"'] & B \arrow[dd, "1"] \\ & \\ B \arrow[r, "1"] \arrow[d, "x"'] & B \arrow[d, "x"] \\ D \arrow[r, "1"'] & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[d, "x"] \\ C \arrow[r, "m"] \arrow[dd, "m"'] & D \arrow[dd, "1"] \\ & \\ D \arrow[r, "1"'] & D \end{tikzcd} . \tag{5.3}$$

Now we have $yx=1$ by (5.2), while $xy=1$ by (5.3) and the uniqueness clause in the definition of a reflexion.

COROLLARY 5.2. *When $\mathscr{M}$, besides containing the identities, is replete in $\mathscr{A}^2$, the reflexion into $\mathscr{M}$ of any $f : A \to B$ can be taken to be of the form*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[d, "1"] \\ C \arrow[r, "m"'] & B \end{tikzcd} . \tag{5.4}$$

REMARK 5.3. *For the rest of this section we suppose that $\mathscr{M}$ is a class of morphisms containing the identities and replete in $\mathscr{A}^2$, and we write $\mathcal{E}$ for the class $\mathscr{M}^{\uparrow}$ of morphisms.* Departing somewhat from the language of Tholen [8], we call (q, m) an *$\mathscr{M}$-factorization* of f if $f=mq$ with $m \in \mathscr{M}$ and if $(q, 1) : f \to m$ is, as in (5.4), a reflexion of $f \in \mathscr{A}^2$ into $\mathscr{M}$. Such an $\mathscr{M}$-factorization of f, if it exists, is of course unique up to the replacement of C by an isomorph. The universal property asserting that (5.4) is a reflexion of f into $\mathcal{M}$ can be expressed as follows: $f=mq$ with $m\in\mathcal{M}$, and whenever $nu=vf$ with $n\in\mathcal{M}$, we have a unique w rendering commutative

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "q"] \arrow[dd, "u"'] & & B \arrow[dd, "v"] \\ & C \arrow[ur, "m"] \arrow[dl, "w"] & \\ X \arrow[rr, "n"'] & & Y \end{tikzcd} \tag{5.5}$$

In other words, $f=mq$ with $m\in\mathcal{M}$ and $(q, m)\downarrow n$ for all $n\in\mathcal{M}$. To say that every f admits an $\mathcal{M}$-factorization, or that *$\mathcal{M}$-factorizations exist*, is of course to say that $\mathcal{M}$ is reflective in $\mathcal{A}^2$. When this is the case, the result of Proposition 5.1 is given in MacDonald-Tholen [6] (Proposition 1.3, page 178).

It is not in general the case that the $\mathcal{M}$-factorization (q, m) of f has $q\in\mathcal{E}$; we shall see in Proposition 5.9 below that this is so for all f precisely when $\mathcal{M}$ is closed under composition. Yet in the other direction we do have the trivial result:

PROPOSITION 5.4. *If $f=mq$ with $m\in\mathcal{M}$ and $q\in\mathcal{E}$, then (q, m) is an $\mathcal{M}$-factorization of f.*

PROPOSITION 5.5. *For $f: A\rightarrow B$ the following are equivalent:*
(i) $f\in\mathcal{M}$;
(ii) *f has the $\mathcal{M}$-factorization $(1_A, f)$;*
(iii) *f has an $\mathcal{M}$-factorization (q, m) with q invertible;*
(iv) *f has an $\mathcal{M}$-factorization (q, m) and there exists a t rendering commutative*

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "1"'] & C \arrow[d, "m"] \arrow[dl, "t"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Proof. Obviously (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv). Given (iv), we have commutativity in

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[d, "q"'] & C \arrow[r, "m"] \arrow[dl, "qt"] & B \arrow[d, "1"] \\
C \arrow[rr, "m"'] & & B,
\end{tikzcd}$$

whence $qt=1$ by the uniqueness of w in (5.5). Thus q is invertible, and $f \in \mathcal{M}$.

PROPOSITION 5.6. *For $f : A \to B$ the following are equivalent*:

(i) $f \in \mathcal{E}$;

(ii) *f has the $\mathcal{M}$-factorization* $(f, 1_B)$;

(iii) *f has an $\mathcal{M}$-factorization* (q, m) *with m invertible*;

(iv) *f has an $\mathcal{M}$-factorization* (q, m) *and there exists an s rendering commutative*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "q"'] & B \arrow[d, "1"] \arrow[dl, "s"] \\
C \arrow[r, "m"'] & B
\end{tikzcd}$$

Proof. Obviously (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv). Given (iv), we have commutativity in

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[d, "q"'] & C \arrow[r, "m"] \arrow[dl, "sm"] & B \arrow[d, "1"] \\
C \arrow[rr, "m"'] & & B,
\end{tikzcd}$$

whence $sm=1$ by the uniqueness of w in (5.5). Thus m is invertible; from which it follows easily that $f \in \mathcal{E}$.

We can now give a "partial converse" to Proposition 4.3:

PROPOSITION 5.7. *If $\mathcal{M}$ is reflective in $\mathcal{A}^2$, then $\mathcal{M}=\mathcal{F}^{\downarrow}$ where $\mathcal{F}$ is the class of all $\mathcal{M}$-factorizations of morphisms.*

Proof. We have $\mathcal{M}\subset\mathcal{F}^{\downarrow}$ by Remark 5.3. Let $f\in\mathcal{F}^{\downarrow}$ and let f have the $\mathcal{M}$-factorization (q, m). Then there is a t rendering commutative

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[d, "1"'] & C \arrow[r, "m"] \arrow[dl, "t"] & B \arrow[d, "1"] \\
A \arrow[rr, "f"'] & & B,
\end{tikzcd}$$

whence $f\in\mathcal{M}$ by Propositon 5.5.

Turning to the case where $\mathcal{A}$ admits pullbacks, we prove an analogue of Proposition 3.3:

PROPOSITION 5.8. *If $f: A\rightarrow B$ admits an $\mathcal{M}$-factorization $f=(q, m)$, then $q: f\rightarrow m$ is a reflexion of $f\in\mathcal{A}/B$ into $\mathcal{M}/B$. Conversely, if $\mathcal{A}$ admits pullbacks and $\mathcal{M}$ is stable under pullbacks, and if $q: f\rightarrow m$ is a reflexion of $f\in\mathcal{A}/B$ into $\mathcal{M}/B$, then (q, m) is an $\mathcal{M}$-factorization of f.*

Proof. The first assertion is trivial; for if $f=nu$ with $n\in\mathcal{M}$, the existence of a unique x rendering commutative

$$\begin{tikzcd}
 & C \arrow[dr, "m"] \arrow[dd, "x"] & \\
A \arrow[ur, "q"] \arrow[dr, "u"'] & & B \\
 & X \arrow[ur, "n"'] &
\end{tikzcd}$$

is a special case of the universal property (5.5). For the other direction we consider $n\in\mathcal{M}$ and u, v as in (5.5) with $nu=vmq$, and construct

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[dr, "t"] \arrow[dd, "u"'] & C \arrow[r, "m"] \arrow[d, "x"] & B \arrow[dd, "v"] \\
 & D \arrow[ur, "r"'] \arrow[dl, "s"] & \\
X \arrow[rr, "n"'] & & Y
\end{tikzcd}$$

where D is the pullback of n and v, t is the unique morphism with $rt=mq$ and $st=u$, and x is the unique morphism with $xq=t$ and $rx=m$, which exists by hypothesis since $r\in\mathcal{M}$. To obtain the universal property (5.5), it remains to show that any $w: C\to X$ with $wq=u$ and $nw=vm$ is in fact sx. Given such a w there is, since D is the pullback, some $y: C\to D$ with $sy=w$ and $ry=m$. Now $wq=u$ gives $syq=u$, while $ryq=mq$; whence, since D is the pullback, $yq=t$. This, along with $ry=m$, gives $y=x$, so that $w=sy=sx$.

The following results on the special case where $\mathcal{M}$ is closed under composition are contained partly in [6] (Proposition 2.1, page 68) and partly in [8] (Proposition 1.2, page 177), and we do no more than organize them for the reader's convenience.

PROPOSITION 5.9. *When $\mathcal{M}$-factorizations exist the following are equivalent*:

(i) *Every $\mathcal{M}$-factorization (q, m) has $q\in\mathcal{E}$.*

(ii) *Every $\mathcal{M}$-factorization (q, m) with $q\in\mathcal{M}$ has q invertible.*

(iii) *$\mathcal{M}$ is closed under composition.*

Proof. (i) implies (ii) since it is trivial (see [2] Proposition 2.1.2, page 173) that $\mathcal{M}\cap\mathcal{E}=\mathcal{M}\cap\mathcal{M}^{\dagger}$ consists of isomorphisms. To see that (ii) implies (iii), let (q, m) be the $\mathcal{M}$-factorization of nk where $n, k\in\mathcal{M}$, and let t be the unique morphism rendering commutative

$$\begin{tikzcd}
 & D \arrow[dr, "n"] & \\
A \arrow[ur, "k"] \arrow[dr, "q"] \arrow[dd, "k"'] & & B \arrow[dd, "1"] \\
 & C \arrow[ur, "m"'] \arrow[dl, "t"] & \\
D \arrow[rr, "n"'] & & B.
\end{tikzcd}$$

The M8 part of Theorem 2.5 gives $t\in\mathcal{M}$ since $n, m\in\mathcal{M}$, and then $q\in\mathcal{M}$ since $t, k\in\mathcal{M}$. So q is invertible by (ii), whence $nk=mq\in\mathcal{M}$. To see that (iii) implies (i), suppose that the q of (i) has the $\mathcal{M}$-factorization (r, n). Since $mn\in\mathcal{M}$ by (iii), we have an s rendering commutative

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[d, "r"'] & C \arrow[r, "m"] \arrow[dl, "s"] & B \arrow[d, "1"] \\
D \arrow[r, "n"'] & C \arrow[r, "m"'] & B;
\end{tikzcd}$$

but now the uniqueness of w in (5.5) gives $ns=1$, whence $q\in\mathcal{E}$ by Proposition 5.6.

Recall from [2] that a *factorization system* on $\mathscr{A}$ consists of two classes $\mathcal{E}$, $\mathcal{M}$ of morphisms, each containing the identities and closed under composition, such that $e\downarrow m$ whenever $e\in\mathcal{E}$ and $m\in\mathcal{M}$, and such that $(\mathcal{E},\mathcal{M})$ *factorizations exist*, in the sense that every morphism f admits a factorization $f=me$ with $m\in\mathcal{M}$ and $e\in\mathcal{E}$. Recall further from [2] that every factorization system $(\mathcal{E},\mathcal{M})$ is a prefactorization system, and that a prefactorization system $(\mathcal{E},\mathcal{M})$ is a factorization system precisely when $(\mathcal{E},\mathcal{M})$ factorizations exist.

THEOREM 5.10. *Let $\mathcal{M}$ be any class of morphisms in $\mathscr{A}$, and set $\mathcal{E}=\mathcal{M}^{\uparrow}$. Then the following are equivalment.*
(i) *$(\mathcal{E},\mathcal{M})$ is a factorization system.*
(ii) *$\mathcal{M}$ is a replete reflective subcategory of $\mathscr{A}^2$ containing the identities and closed under composition.*

Proof. Given (i), $\mathcal{M}=\mathcal{E}^{\downarrow}$ contains the identities, is replete in $\mathscr{A}^2$, and is closed under composition, by Proposition 4.3 and the M4 part of Theorem 2.5. If $f=me$ with $m\in\mathcal{M}$ and $e\in\mathcal{E}$, clearly (e,m) is an $\mathcal{M}$-factorization of f, so that $\mathcal{M}$ is reflective. Given (ii), $\mathcal{M}$ is closed under composition and contains the isomorphisms by hypothesis, while the same is true of $\mathcal{E}=\mathcal{M}^{\uparrow}$ by (the dual of) Proposition 4.3 and the M6 part of Theorem 2.5. Trivially $e\downarrow m$ whenever $e\in\mathcal{E}$ and $m\in\mathcal{M}$ since $\mathcal{E}=\mathcal{M}^{\uparrow}$. By Proposition 5.9, the $\mathcal{M}$-factorization (q,m) of any morphism f has $q\in\mathcal{E}$. Hence $(\mathcal{E},\mathcal{M})$ is a factorization system.

6. Examples of reflective $\mathcal{M}$, and remarks

EXAMPLE 6.1. The authors of [6] consider the example where $\mathscr{A}$ is the category of categories and right-adjoint functors, while $\mathcal{M}$ is the class of monadic functors. If t is the monad on B derived from the right adjoint $f : A \to B$, let B^t be the category of algebras, $m : B^t \to B$ the forgetful functor, and $q : A \to B^t$ the comparison functor; then (q, m) is an $\mathscr{M}$-factorization of f if A and B^t admit coequalizers. If we restrict $\mathscr{A}$ to consist of cocomplete categories and right adjoints with rank, then all $\mathscr{M}$-factorizations exist. Here $\mathscr{M}$ is not closed under composition, and does not consist of monomorphisms.

EXAMPLE 6.2. When $\mathscr{M}$, containing the identities and replete in $\mathscr{A}^2$, does consist of monomorphisms, it is certainly reflective in $\mathscr{A}^2$ if pullbacks of $\mathscr{M}$'s and arbitrary (even large) intersections of $\mathscr{A}$'s exist, and these again lie in $\mathscr{M}$; for by Proposition 5.8 we have only to give a reflexion of $f : A \to B$ into $\mathscr{M}/B$, and we find this as the intersection $m : C \to B$ of those $n : D \to B$ in M through which f factorizes. This is Theorem 2.4 of Tholen [8], who points out that even more is true: in this case any *family* $(f_i : A_i \to B)$ has an $\mathscr{M}$-factorization (q_i, m), the extension of the notion of $\mathscr{M}$-factorization to families being the obvious one. When $\mathscr{M}$ does not consist of monomorphisms, the hypothesis that arbitrary fibred products of $\mathscr{M}$'s exist cannot hold: see Section 1.3 of [5]. For results on the reflectivity of an $\mathscr{M}$ not consisting of monomorphisms, see Tholen [8].

EXAMPLE 6.3. In particular, by Proposition 4.2 and Theorem 2.5, the class $\mathscr{M}$ of regular monomorphisms is reflective in $\mathscr{A}^2$ if $\mathscr{A}$ admits pullbacks and arbitrary intersections of regular monomorphisms. However it is not clear that the $\mathscr{M}$-factorization (q, m) of a morphism f is then what was called in [4] (in the dual case) the *regular factorization* of f. The latter was defined as a pair (q, m) with $f = mq$ such that m is the joint equalizer of all pairs x, y with $xf = yf$. The reader will easily verify (this is Example 2.1 of [6]) that a regular factorization is always an $\mathscr{M}$-factorization, while an $\mathscr{M}$-factorization is a regular factorization if $\mathscr{A}$ admits equalizers. But then we do not need pullbacks in $\mathscr{A}$: by [4] (Proposition 4.2 on page 134), if $\mathscr{A}$ admits equalizers, regular factorizations exist if $\mathscr{A}$ admits either arbitrary intersections of regular monomorphisms or else pushouts.

REMARK 6.4. The most classical examples of a reflective $\mathscr{M}$ are given as in Theorem 5.10 by a factorization system.

REMARK 6.5. The authors of [6] consider transfinite iteration of $\mathscr{M}$-factorizations, where $\mathscr{M}$ contains the identities and is replete and reflective in $\mathscr{A}^2$, but do not go beyond small ordinals. When $\mathscr{M}$ consists of monomorphisms [resp. strong monomorphisms] and $\mathscr{A}$ admits all intersections of monomorphisms [resp. strong monomorphisms] we can carry this process to its limit and arrive at a factorization system. Beginning with the $\mathscr{M}$-factorization $f=mq=n_0q_0$ of f, we define inductively a factorization $f=n_\alpha q_\alpha$ for each ordinal α; if $\alpha=\beta+1$ we take the $\mathscr{M}$-factorization $q_\beta=m_\beta q_\alpha$ of q_β and set $n_\alpha=n_\beta m_\beta$; if α is a limit-ordinal we set $n_\alpha=\bigcap_{\beta<\alpha}n_\beta$, with the obvious q_α. If we suppose every category to be small with respect to some universe, this process ultimately terminates; which means that m_β is invertible for some β, and hence by Proposition 5.6 that $q_\beta\in\mathscr{E}$. By Proposition 4.3 and Theorem 2.5, we have $n_\beta\in\mathscr{E}^\downarrow$. Thus the prefactorization system $(\mathscr{E},\mathscr{E}^\downarrow)=(\mathscr{M}^\uparrow,\mathscr{M}^{\uparrow\downarrow})$ is actually a factorization system. When $\mathscr{M}$ consists of the regular monomorphisms and $\mathscr{A}$ admits equalizers, this gives (see [4], Proposition 3.9 on page 133) the factorization of f into an epimorphism followed by a strong monomorphism.

References

1. A.K. Bousfield, *Construction of factorization systems in categories*, J. Pure Appl. Algebra **9**(1977), 207–220.
2. P.J. Freyd and G.M. Kelly, *Categories of continuous functors I*, J. Pure Appl. Algebra **2**(1972), 169–191.
3. G.B. Im and G.M. Kelly, *Some remarks on conservative functors with left adjoints*, J. Korean Math. Soc. **23**(1986), 19–33
4. G.M. Kelly, *Monomorphisms, epimorphisms, and pull-backs*, J. Austral. Math. Soc. **9**(1969), 124–142.
5. G.M. Kelley, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bull. Austral. Math. Soc. **22**(1980), 1–83.
6. J. MacDonald and W. Tholen, *Decomposition of morphisms into infinitely many factors*, Lecture Notes in Math. **962**(1982), 175–189.
7. C.M. Ringel, *Diagonalisierungspaare* I, Math. Z. **117**(1970), 249–266.
8. W. Tholen, *Factorizations, localizations, and the orthogonal subcategory problem*, Math. Nacher. **114**(1983), 63–85.

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