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A note on C–Closed and Compactification

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C– Closed와 Compactification에 관하여

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國 文 要 約

이 論文에서는 net와 連續函數의 族에서 생성된 像에 의한 Compact 空間의 性質을 조사하였다. Compact 空間에 기초를 둔 C–closed, A–net, A–變換을 사용하여 Haussdorff 空間 혹은 Regular 空間에서의 그들의 性質을 살펴보았다.

I. INTRODUCTION

On this paper, the characterization of a Compact space by the nets and the images induced by any family of continuous function. Using the C–closed, A–transformation based on the Compactification, investigate the their properties on Hausdorff space or Regular space.

II. PRELIMINARY

DEFINITION 1)

Let A be a family of continuous on continuous functions on topological space X. A net $\{X_i\}$ in X will be net an A–net, if $\{f(X_i)\}$ converges for each f in A.

DEFINITION 2)

Let $A=\{f_a \mid \alpha\in\wedge\}$ be a family of continuous functions f_a on topological space X into Haussdorff space X_a. Let S be the collection of all A–nets in X. Define δ on S setting that $\{x_i\}$ δ $\{y_i\}$ if and only if $\lim\{f_a(x_i)\}=\lim\{f_a(y_i)\}$ for each f_a in A.

DEFINITION 3)

A– transformation (Y, e) of X if and only if $e: X\rightarrow Y$ and $e(x)$ is the equivalence class in Y containing a net x in X.

DEFINITION 4)

A space X is called C–closed if every countably compact subset of X is closed in X. Equivalently X is C–closed if every non–closed subset A of contains a sequence x_n which has no cluster point in A.

TYCHONOFF THEOREM]

Let $\Pi_{\alpha\in\wedge} X\alpha$ be the product space of $\{X\alpha \mid \alpha\in\wedge\}$, A a collection of projection maps of $\Pi_{\alpha\in\wedge} X\alpha$ onto $X\alpha$. Since every A-net in $\Pi_{\alpha\in\wedge} X\alpha$ converges in $\Pi_{\alpha\in\wedge} X\alpha$ is compact if and only if $X\alpha$ is compact for each α in $\wedge$.

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BANACH - ALAOGLU THEOREM]

Let E be a topological vector space, E^* the continuous dual of E, and $\sigma(E^*, E)$ the weak topology on E^* induced by E. For every neighborhood U of N in E, let U^0 be the polar of U in E^*. Then U^0 is a closed subset in $(E^*, \sigma(E^*, E))$ satisfying that cl $(x(U^0))$ is compact for each x in E and every E-net $\{y_i\}$ in U^0, where

$y(x) = \lim \{y_i(x)\}$ for all x in E.

Ⅲ. PROPOSITIONS AND THEOREMS

PROPOSITION 1]

δ is an equivalence relation.

PROPOSITION 2]

Let $Z = S/\delta$ be the collection of all equivalence classed in S. An equivalence class in Z containing an A-net $\{x_i\}$. (It will be denoted by $\{\bar{x}_i\}$.) For each $f\alpha$ on Z by setting that $f\alpha(\{\bar{x}_i\}) = \lim \{f\alpha(\bar{x}_i)\}$. The $A = \{f\alpha \mid \alpha \in \Lambda\}$ seperates points of Z ; i.e.for any two points z_1, z_2 in Z, there is an $f\alpha$ in A such that $f\alpha(z_1) \neq f\alpha(z_2)$. Each $f\alpha$ will be called the extension of $f\alpha$ to Z.

PROPOSITION 3]

Let $A = \{f\alpha \mid \alpha \in \Lambda\}$ be a family of continuous functions $f\alpha$ on X into Hausdorff space $X\alpha$ such that the topology on X is the weak topology induced by A, E, F two subspaces of X such that $E \subset F \subset \mathrm{cl}(E)$. (cl$(E)$ is the closure of E). Then the following are equivalent:

1) Every A-net in E has cluster point in F

2) Every A-net in F converges in F.

PROOF]

Let $\{y_i\}$ be an A-net in F. For each y_i, pick a net $\{z_{ij}\}$ in E converging to y_j. for each $f\alpha$ in A, $\{f\alpha(y_i)\}$ converges to a point z_α and $\lim\{f\alpha(x_{ij})\} = f\alpha(y_i)$, so that for each neighborhood $N\alpha$ of z_α, there is an x_α in E such that $f_\alpha(x_\alpha) \in N_\alpha$. Thus for any finite subset H of A and $N_H = \{N_\alpha \mid N_\alpha$ is an open neighborhood of z, $\alpha \in H\}$, there is an x_H in E such that $f_\alpha(x_H)$ is in N_α for each α in H. Direct $\{(H, t) \mid H$ is a finite subset of A, N_α is an open neighborhood of z, $\alpha \in H\}$ by setting that $(H_1, t_1) \geqq (H_2, t_2)$ if and only if $H_1 \subset H_2$ for each a in H_2. Then $\{x_H\}$ is an A-net in E. 1) implies that $\{x_H\}$ has a cluster point x in E. Since X is Haussdorff and $\{f_\alpha(x_H)\}$ converges for each α in A, so that $\{f_\alpha(x_H)$ converges to $f_\alpha(x)$ for each α in A. This implies that $\lim\{f_\alpha(y_i)\} = f_\alpha(x)$ for each α in A. thus $\{y_i\}$ converges to x. 2) implying 1) in obvious.

PROPOSITIONS 4]

Every *Haussdorff* sequential space is C-closed.

PROPOSITIONS 5]

If X is a *Haussdorff* space and every countaly compact subset of X is sequential, then X is C-closed.

PROPOSITION 6]

If a *Haussdorff* space X admits a continuous one-to-one map into a C-closed space, then X is C-closed.

PROOF]

Let $f: X \to Y$ be a continuous one-to-one map, where Y is C-closed. Let A be a countably compact subset of X. Then $f(A)$ is cauntably compact subset of Y. Since Y is C-closed, $f(A)$ is closed in Y. Therefore $f^{-1}(f(A)) = A$ is closed in X.

THEOREM 1]

If in a ***regular*** space X, every point has a C-closed neighborhood, then X is C-closed.

PROOF]

Let A be a countably compact subset of X and $x \in \mathrm{cl}(X)$. we have claim that $x \in A$. Let U be an open subset of X such that $x \in U$ and U is C-closed. By regularity, we choose an open set V such that $x \in V$ and $\mathrm{cl}(V) \subset U$. Since $\mathrm{cl}(V) \cap A$ is countably compact subset of U, it is closed in U. But $x \in \mathrm{cl}(\mathrm{cl}(V) \cap A)$. Therefore $x \in A$. This shows that A is closed in X.

THEOREM 2]

Let A be any family of continuous functions on a topological space X. then X is compact if and only if

1) $f(X)$ is contained in a compact subset C for each f in A

2) every A-net has a cluster point in X.

PROOF]

⇦) Let x be an ultranet in X. for each f in A, $\{f(x_i)\}$ is an ultranet in C, hence converges in C; that is, $\{x_i\}$ is an A-net. 2) implies that $\{x_i\}$ has a cluster point x in X. Since $\{x_i\}$ is an ultranet, $\{x_i\}$ converges to x. Thus, X is compact.

⇨) The converse is easily proved.

COROLLARY]

Let A be a family of continuous functions on X into *Hausdorff* spaces such that the topology on X is the weak topology induced by A, E, F two subspaces of X such that $E \subset F \subset \mathrm{cl}(E)$. then F is compact if and only if

1) $\mathrm{cl}(f(E))$ is compact for each f in A

2) every A-net in E converges in F.

PROPOSITION 7]

e is continuous and $e(X)$ is dense in Y.

PROOF]

If $\{x_i\}$ converges to x, then $\lim\{\bar{f}_\alpha(e(x_i))\} = \{f_\alpha(x_i)\} = \bar{f}_\alpha(x) = \bar{f}_\alpha(e(x))$ for each $\bar{f}_\alpha$ in $\bar{A}$. implying that $\{e(x_i)\}$ converges to $e(x)$. Thus e is continuous. For the density of $e(X)$ in Y, let $\{\bar{x}_i\}$ be in Y, then $\lim\{f_\alpha(e(x_i))\} = f(\{\bar{x}_i\})$ for each $\bar{f}_\alpha$ in $\bar{A}$, implying that $\{e(x_i)\}$ converges to $\{\bar{x}_i\}$.

COROLLARY]

Let $\{\bar{x}_i\}$ be in Y, then $\{e(x_i)\}$ converges to $\{\bar{x}_i\}$ in Y.

THEOREM 3]

e is an embedding of X as a dence subspace of Z if and only A separates points of X and the topology on X is the weak topology induced by A.

PROOF]

For Z and e, previous defined, e is one-to-one if and only if A separates of X. Since for each x in X, $\bar{f}_\alpha(e(x)) = f_\alpha(x)$ for each f in A, so that e is an open mapping from X onto $e(X)$ if and only if the topology on X is the weak topology induced by A. Thus PROPOSITION 2] implies this theorem.

PROPOSITION 8]

Every A-net $\{y_i\}$ in Y converges in Y.

PROOF]

For each y_i in $\{y_i\}$, let x be an A-net in X belonging to y_i. For each f_α in A, $\{\bar{f}_\alpha(y_i)\}$ converges to a point z and $\lim\{f(x_{ij})\} = f_\alpha(y_i)$, so that for every open neighborhood N_α of z_α, there is an x_α in X such that $f_\alpha(x_\alpha)$ is in N_α. By the same techniques in the proof of PROPOSITION 1] we can get an A-net $\{x_H\}$ in X such that $\bar{f}_\alpha(\{x_H\}) = \lim\{f_\alpha(x_H)\} = z = \lim\{f_\alpha(y_i)\}$. Thus $\{y_i\}$ converges to $\{x_H\}$.

PROPOSITION 9]

Y is compact if and only if $\mathrm{cl}(f(X))$ is compact for each f in A.

PROOF]

By the COROLLARY of THEOREM 1 and previous PROPOSITION and the way constructing Y, Y is compact if and only if $cl(f_\alpha(Y))$ is compact for each f in A. Since $cl(f_\alpha(Y))$ is compact for each f_α in A, thus the statement follows.

THEOREM 4]

(Y, e) is a compactification of X and only if.

1) for each f_α in A, $cl(f_\alpha(X))$ is compact,

2) A separates points of X.

3) the topology on X is the weak topology induced by A.

PROOF]

This theorem follows immediately THEOREM 3] and PROPOSITION 9].

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