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On Completely Reducible Modules

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1. Introduction and Preliminaries

In this paper we will investigate some properties of completely reducible module. First of all, we recall the notion of completely reducible modules.

Definition 1.1. A left R-module M is said to be completely reducible if every submodule is a direct summand.

Lemma 1.2. *A submodule of a completely reducible module is completely reducible.*

Definition 1.3. Let R be a ring and $\{M_i | i \in I\}$ a family of R-modules. The complete direct sum $\sum_c M_i$ consists of the functions m defined on I with values $m(i) \in M_i$.

Definition 1.4. Let R be a ring with identity. An R-module U is homogeneous of type I if U is the direct sum of a family $\{M_\lambda | \lambda \in \Lambda\}$ of irreducible R-submodules M_λ, each isomorphic to the irreducible module I.

2. Main Theorem

Theorem 1.5. *An artinian ring identity is semisimple if and only if each unital R-modules is completely reducible.*

Proof. Since R is artinian, this entails showing that there exist no nonzero nilpotent left ideals. By hypothesis, given a left ideal $L \neq (0)$, there exists L' such that $R = L \oplus L'$. In particular $1 = e + e'$ with $0 \neq e \in L$. It follows that $e = e^2 + ee'$ and hence $e - e^2 \in L' \cap L = (0)$. Therefore $e = e^2$. Since $0 \neq e = e^2 \in L$. L is not nilpotent.

Theorem 1.6. *Let* 1 *R and Let M be a unital, completely reducible R-module K its centralizer. If $m \neq 0$ belongs to an irreducible submodule U of M, mK is an irreducible K-module.*

Proof. Let $0 \neq m' \in mK$. Then there exists $\alpha \in K$ such that $m' = m\alpha$. The mapping $u \longrightarrow u\alpha$ of U onto $U\alpha$ is an isomorphism. This may be extended to an automorphism β of M. Therefore $m' = m\beta$ and hence $m'\beta^{-1} = m$ showing that $m'K = mK$.

Theorem 1.7. *The following three statements about an R-module M are equivalent*

(1) *M is completely reducible.*

(2) *M is a direct sum of irreducible submodules.*

(3) *M is a sum of irreducible submodules.*

Proof. (1) implies (2).

Let $n \in N$, $n \neq 0$ and N is submodule of M. Consider the collection of all submodules

N' of N such that $n \notin N'$. Since N is a submodule of M, we know that N is completely reducible by lemma 1. 1, and so $N = N_0 \oplus N_1$ for some submodule N_1 of N and we may show that N_1 is irreducible. For otherwise N_1 would contain a proper nonzero submodule N_2, and then $N_1 = N_2 \neq N_3$ for some nonzero submodule N_3 of N_1. But this gives $N = N_0 \oplus N_2 \oplus N_3$ and surely either $n \notin N_0 + N_2$ or $n \notin N_0 + N_3$, since $(N_0 + N_2) \cap (N_0 + N_3) = N_0$ This shows that N_1 is irreducible.

(2) implies (3)

The fact that (2) implies (3) is immediate.

(3) implies (1) .

Let N be a submodule of M, N' be a submodule maximal with respect to the property that $N' \cap N = (0)$. We wish to prove that $N \oplus N' = M$ for by construction. The sum $N + N'$ is direct. Suppose the result is false. Then there exists m in M such that $m \notin N + N'$. By (3), $m = m_1 + \cdots + m_s$ where the $\{m_i\}$ belong to irreducible submodules $\{M_i\}$. Since $m \notin N + N'$, some $m_i \notin N + N'$, and there exists an irreducible submodule M_i such that $M_i \not\subset N + N'$. Because M_i is irreducible we have $M_i \cap (N + N') = (0)$ and hence $N' + M_i$ is a submodule properly containing N' whose intersection with N is zero. This contradicts the maximality of N' and we must have $N + N' = M$.

Corollary 1.8. Let the R-module M satisfy either chain condition. Then M is completely reducible if and only if M is a direct sum of a finite set of irreducible submodules.

Theorm 1.9. *Let G be a finite group and K a field whose characteristic does not divide $[G:1]$. Then every left KG-module is completely reducible.*

Proof. see[15 p. 88]

Reference

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