Spectral Theorem and its Application

by Dong Jin Yoo

*Kyung Hee University, Seoul, Korea*

**Introduction**

In the standpoint of engineering applications, eigenvalue problems among the most important problems in connection with matrices, and the number of research papers on developing numerical methods for computers is enormous. The basic concepts are as follows.

Let \( A = (a_{ij}) \) be a given square \( n \)-rowed matrix and consider vector equation

\[ \lambda X = \Sigma a_{ij} X = O \]  

where \( \lambda \) is a number.

The value of \( \lambda \) for which (1) has a solution \( X \neq O \) is called an eigenvalue or characteristic value of the matrix \( A \).

The corresponding solutions \( X \neq O \) of (1) are called eigenvectors or characteristic vectors of \( A \) corresponding to that eigenvalue \( \lambda \).

The set \( \{\lambda_i\} \) is called the spectrum of \( A \).

The spectral radius of \( A \) is called the spectral radius of \( A \).

Problems of this type occur in connection with physical and technical applications.

**Theorem**

Assume \( A \) is a compact self adjoint operator on \( H \) (Hilbert space). There exist an orthonormal system \( \varphi_1, \varphi_2, \ldots \) of eigenvectors of \( A \) and corresponding eigenvalues \( \lambda_1, \lambda_2, \ldots \) such that for all \( X \in H \),

\[ \langle X, \varphi_n \rangle = \lambda_n \langle X, \varphi_n \rangle \quad \text{or} \quad \sum_{k=1}^{n} \lambda_k < X, \varphi_k > \varphi_k. \]

\( \{\lambda_n\} \) is an infinite sequence, then it converges to zero.

\( \{\varphi_n\} \) is an orthonormal basis for \( H \), then the matrix corresponding to \( A \) and \( \{\varphi_n\} \) is a diagonal matrix.

**Application**

If \( (a_{ij}) \) is a square \( n \)-rowed matrix, \( AX = \lambda X \).

The set \( \{\lambda_k\} \) of the eigenvalues is called the spectrum of \( A \).

The largest of the absolute values of the eigenvalues of \( A \) is called the spectral radius of \( A \).

\[ \frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \] : the vibrations of an elastic string, where \( u(x, t) \) is deflection of the string.
\( x = 0, \ x = l \): fixed at the ends.

Boundary condition: \( (2) \ u(0, t) = 0, \ u(l, t) = 0 \) for all \( t \).

Initial condition: \( (3) \ u(x, 0) = f(x), \ (4) \ \left[ \frac{\partial u}{\partial t} \right]_{t=0} = g(x) \).

The form \( u(x, t) = F(x) G(t) \): solution of \( (1) \)

\[
\frac{\partial^2 u}{\partial t^2} = F G'', \quad \frac{\partial^2 u}{\partial x^2} = F'' G \Rightarrow F G'' = C^2 F G = \frac{F''}{G} = K.
\]

\[ \therefore \ (a) \ F'' - KF = 0 \quad (b) \ G'' - C^2 K G = 0 \]

From \( (2), \ u(0, t) = F(0) G(t), \ u(l, t) = F(l) G(t) = 0 \) for all \( t \).

(a) \( F'' - KF = 0 \) : For \( K = \mu^2, \ F = A e^{\mu x} + B e^{-\mu x}, \ K = -\mu^2 \Rightarrow F'' + \mu^2 F = 0 \)

\( F(x) = A \cos \mu x + B \sin \mu x \): general solution.

\( F(0) = A = 0 \) and the \( F(l) = B \sin \mu l = 0 \) \( (B \neq 0) \)

\( \therefore \sin \mu l = 0, \ \mu l = n \pi \ or \ \mu = \frac{n \pi}{l} \)

Setting \( B = 1, \ F(x) = F_n(x), \ F(x) = \sin \frac{n \pi x}{l} \quad (n = 1, 2, \ldots) \).

(b) \( G'' + \lambda_n^2 G = 0 \) \( (\lambda_n = cn \pi / l) \Rightarrow G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t \): general solution

\( \therefore \ u_n(x, t) = F_n(x) G_n(t) = \sin \frac{n \pi x}{l} \quad (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \quad (n = 1, 2, \ldots) \).

The set \( \{ \lambda_1, \lambda_2, \ldots \} \) is called the spectrum.

\[
\begin{align*}
\lambda_1 & \quad \lambda_2 & \quad \lambda_3 & \quad \ldots \\
\text{vibrating string} & \\
n = 1 & \\
n = 2 & \\
n = 3 & \\
\text{normal mode for various values}
\end{align*}
\]

\( (3) \ u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{l} = f(x) \quad (B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n \pi x}{l} \ dx) \quad (n = 1, 2, \ldots) \).

\( (4) \ \left[ \frac{\partial u}{\partial t} \right]_{t=0} = \left[ \sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n \pi x}{l} \right]_{t=0} = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n \pi x}{l} = g \)

For \( t = 0, \ B_n^* \lambda_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n \pi x}{l} \ dx \).

Since \( \lambda_n = cn \pi / l, \ B_n^* = \frac{2}{cn \pi} \int_0^l g(x) \sin \frac{n \pi x}{l} \ dx \quad (n = 1, 2, \ldots) \).

References

1. Israel Gohberg, *Basic Operator Theory.*