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The Norm-Preserving Linear Maps in Abelian Algebras

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Introduction

$f: A \longrightarrow A$ is a K-algebra automorphism, then f preserves norms. Where A is a -dimensional associative K-algebra with unity. In this paper we shall consider the erse problem in the case when A is a finite-dimensional *semisimple abelian* K-alge- Throughout this paper K is an *infinite* field of an arbitrary characteristic.

Main Theorems

mma 1. *Let F_i be purely inseparable finite extensions of K, $i=1, \cdots, r$. Put* $\bar{}_1 \dotplus F_2 \dotplus \cdots \dotplus F_r$. *Suppose that $f: A \longrightarrow A$ is a K-linear norm-preserving map such* $f(1)=1$. *Then f is a K-algebra automorphism of A.*

'roof. Put $[F_i : K]=n_i$. Then $N_{A/K}(X_1, \cdots, X_r)=X_1^{n_1} \cdots X_r^{n_r}$. Hence the routine degree nent shows that $f(X_1, \cdots, X_r)$ is only the permutation of $(X_1, \cdots, X_r)$. Hence f is a ebra automorphism of A. This completes the proof.

nark. The above proof also shows that $Aut_K(A)$ is a finite group.

nma 2. *Let F/K be a finite extension and let E' be a separable closure of K in et $f(X) \in E'[X]$ be irreducible over E'. Then $f(X)$ is also irreducible over F if is separable.*

'roof. We may assume $\text{char}(K)=p=$prime number. Put $[F : E']=p^m$. Suppose $f(X)$: irreducible in $F[X]$. Then $f(X)=g(X)h(X)$, $\deg(g)$, $\deg(h) \geqq 1$,
, $h(X) \in F[X]$.

nce $(f(X))^{p^m}=(g(X))^{p^m}(h(X))^{p^m}$, $g(X)^{p^m}$, $h(X)^{p^m} \in E'[X]$.

nce $f(X) \mid g(X)^{p^m}$ and $f(X) \mid h(X)^{p^m}$. Let α be a zero of $f(X)$.

en $g(\alpha)=h(\alpha)=0$. Hence $f(X)$ is not a separable polynomial. This is a contradiction. completes the proof.

nma 3. *Let F/K be a finite extension and let E' be a separable closure of K in F 'et E/E' be a finite extension such that E/K is a Galois extension. Then* $\Xi = E_1 \dotplus \cdots \dotplus E_r$ *(E-algebra isomorphism). Where E_i/E is a purely inseparable sion and $r=[E' : K]$*

'roof. Since E/K is a separable extension there exists $a \in E$ such that $E=K(a)$. Put $=\text{Irr}(a, K, X)$. Then $p(X)=p_1(X) \cdots p_s(X)$, where $p_i(X)$ is a distinct irreducible omial in $E'[X]$ because $p(X)$ is a separable polynomial. $p_i(X)$ is also irreducible over F by Lemma 2.

Now we have the commutative diagram.

$$
\begin{array}{ccc}
E'\otimes_K E & \longrightarrow & F\otimes_K E \\
(E'\text{-algebra isomorphism}) \ \| \wr & & \| \wr \ (F\text{-algebra isomorphism}) \\
E'[X]/(p(X)) & \longrightarrow & F[X]/(p(X)) \\
(E'\text{-algebra isomorphism}) \ \| \wr & & \| \wr \ (F\text{-algebra isomorphism}) \\
\frac{E'[X]}{(p_1(X))} \dot{+} \cdots \dot{+} \frac{E'[X]}{(p_s(X))} & \longrightarrow & \frac{F[X]}{(p_1(X))} \dot{+} \cdots \dot{+} \frac{F[X]}{(p_s(X))}
\end{array}
$$

The above diagram shows that every idempotent in $F\otimes_K E$ belongs to $E'\otimes_K E$ a $F\otimes_K E$ has no nilpotent elements except zero. Therefore $F\otimes_K E\cong E_1\dot{+}E_2\dot{+}\cdots\dot{+}E_r, E_i$, finite extension by the Wedderburn Structure Theorem. Since every idempotent in $F\otimes$ belongs to $E'\otimes_K E$ we can conclude $r\leqq dim_E(E'\otimes_K E)=[E':K]$.

On the other hand $E'=K(b)$ since E' is a separable extension of K.
Put $f(X)=Irr\,(b, K, X)$. Then $f(X)$ splits over E since E is a normal extension of
Therefore $E'\otimes_K E\cong E[X]/(f(X))\cong \underbrace{E\dot{+}E\dot{+}\cdots\dot{+}E}_{r'=[E':K]\text{-times}}$ (E-algebra isomorphism).

Therefore there exists a E-basis $\{b_1, b_2, \cdots, b_{r'}\}$ of $E'\otimes_K E$ such that $b_i^2=b_i$ and b_ib_j for $i\neq j$.

Now consider the isomorphism ϕ: $F\otimes_K E=E_1\dot{+}E_2\dot{+}\cdots\dot{+}E_r$.

Then $(\phi(b_i))^2=\phi(b_i)$ and $\phi(b_i)\cdot\phi(b_j)=0$ for $i\neq j$.

Therefore we conclude that $r'=r$ and $\phi(E'\otimes_K E)=\underbrace{E\dot{+}E\dot{+}\cdots\dot{+}E}_{r\text{-times}}$.

Now $x^{[F:E']}\in E'\otimes_K E$ for all $x\in F\otimes_K E$. Hence E_i/E is a purely inseparable extens This completes the proof.

Theorem 1. *Let A be a finite-dimensional semi-simple abelian K-algebra and let $f: A\to A$ be a norm-preserving K-linear map such that $f(1)=1$. Then f is a K-a morphism of A.*

Proof. By the Wedderburn Structure Theorem we have the K-algebra isomorp $A\cong F_1\dot{+}F_2\dot{+}\cdots\dot{+}F_n$, F_i/K is a finite extension. We may assume that F_i is containe some fixed algebraic closure of K. Let $E_i=K(b_i)$ be a separable closure of K i and let E be the normal closure of $K(b_1, b_2, \cdots, b_n)$. Then
$A\otimes_K E\cong(F_1\otimes_K E)\dot{+}(F_2\otimes_K E)\dot{+}\cdots\dot{+}(F_n\otimes_K E)\cong E_1\dot{+}E_2\dot{+}\cdots\dot{+}E_r$ (E-algebra isomorphisr where E_i/E is a purely inseparable extension by Lemma 3.

Since K is an infinite field $f\otimes_K id$: $A\otimes_K E\longrightarrow A\otimes_K E$ is a E-linear norm-preser map such that $(f\otimes_K id)(1)=1$. Therefore $f\otimes_K id$: $A\otimes_K E\longrightarrow A\otimes_K E$ is an E-algebra tomorphism of $A\otimes_K E$ by Lemma 3. Hence f: $A\longrightarrow A$ is a K-algebra automorphisn This completes the proof.

emark. The above proof also shows that $\mathrm{Aut}_K(A)$ is a finite group.

ounterexample to Theorem 1. Let $K=Z/2Z$, $A=GF(2^n)$. Then every invertible K-
:ar map $f: A \longrightarrow A$ such that $f(1)=1$ preserves norms. Put $G=\{f: A \longrightarrow A \mid f$ is K-
ar and invertible and $f(1)=1\}$. Then $\# G \gneqq (n-1)! \geqq n = \# Aut_K(A)$ for $n \geqq 4$.
:refore Theorem 1. does not hold in this case.

References

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