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Reduced Modified Nodal Approach and Its Zero Diagonal Problem

(縮少變形 노우드解析方式과 그 零對角線問題)

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要 約

본 논문에서는, 축소변형 노우드 해석방식(RMNA)이라고 불리우는 새로운 형태의 회로해석 방정식 형성 방법을 제안한다. 축소변형 노우드해석 방식은 노우드 전압과 제어전류들만을 회로변수로 사용하여, 특수한 형태의 KCL 방정식과 전압전원소자 관계식들을 형성한다. 축소변형 노우드해석 방식은 회로 구조를 고려한 세가지 형태의 피보팅 벡터(pivoting vector)들을 이용하여 주어진 회로에서 부터 직접 그리고 체계적으로 회로해석 방정식들을 형성한다. 즉, 축소 변형 노우드해석 방식의 회로방정식들은 세가지 형태의 선택 벡터들과 함께 소자 도표(element stamp)를 이용하여 쉽게 형성된다. 축소변형 노우드해석 방식의 회로행렬의 크기는 단락 소자가 부가된 회로내의 노우드 수와 동일하며, 그 회로 행렬내의 중심항들은 항상 0이 아닌 값들을 가지게 된다. 축소변형 노우드해석 방식은 노우드해석 방식과 변형 노우드해석 방식의 간단성 및 그 밖의 잇점들을 그대로 유지하면서 문제점들을 효과적으로 해결하고 있다. 특히, 본 논문에서 제안된 변형 노우드회로 해석 방식은 컴퓨터를 이용한 회로해석 프로그램에 사용된다.

Abstract

In this paper, a new method of circuit equation formulation called "Reduced Modified Nodal Approach" (RMNA) is proposed, in which node voltages and controlling currents are adopted as the circuit variables. The circuit equations of the RMNA are formulated directly and systematically from a given circuit through three types of pivotings. In the RMNA matrix, its size is equal to the number of nodes in the circuit augmented by the short-circuit branches, and zero diagonal entries are avoided. The RMNA retains the simplicity and other advantages of the nodal approach and of the MNA, while removing their difficulties.

I. Introduction

The modified nodal approach (MNA) [1] has been widely used for formulating circuit equations in many computer-aided circuit analysis and design programs. This approach is based on the classical nodal approach, but includes circuit elements like voltage sources (VS's) and/or other elements whose currents are required as the circuit variables. This is done by introducing the currents through these branches as additional circuit variables and the corresponding branch relations as additional circuit equations. As a result, the total number of variables is increased and zero diagonal entries may appear in the resultant MNA matrix, which complicates the computer implementation and increases the execution time of the MNA.

Several methods to overcome these difficulties have been proposed in [1-4] Among these, [1] proposes interchange of rows in order to avoid the zero diagonal entries, while

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the method in [2] involves partitioning and ordering the circuit variables. However, they do not consider reduction of circuit variables. In contrast, [3] uses the concept of component cutsets for special circuits such as switched capacitor circuits to reduce the number of circuit variables. In [4], the two-graph modified nodal formulation is developed by the use of separate voltage and current graphs to eliminate redundant variables, but it is difficult to perform pivotings at the formulation stage to avoid zero diagonal entries.

In this paper we develop a new type of nodal approach which solves all the abovementioned problems. The resultant formulation is called the reduced modified nodal approach (RMNA) in the sense that redundant circuit variables in the MNA are effectively eliminated. And three types of pivotings are devised to avoid zero diagonals in its matrix form and to reduce the number of nonzero entries in its upper triangular matrix. As will be seen, the RMNA retains the simplicity and other advantages of the nodal approach and of the MNA, while removing their limitations with little effort.

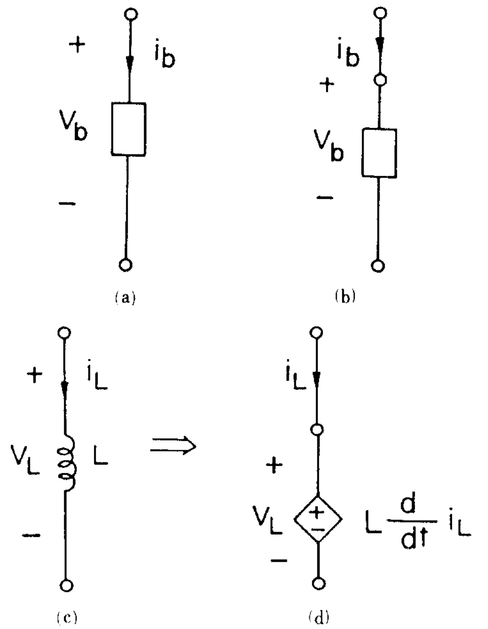


Fig. 1. Insertion of a SCB. (a) Original branch. (b) Modified branch. (c) Treatment of an inductor (as an example of the impedance type element).

II. Reduced Modified Nodal Approach

For a given circuit Np, consider those branches whose currents are required as circuit variables, that is, the controlling branches of current-controlled elements (including impedance type elements) or the branches for output currents of interest. Here, an impedance type element implies one whose voltage is controlled by its own current, such as a current-controlled element described by an equation of the form $v = f(i)$ and the inductor. In the MNA, these branch currents, as well as the branch currents for VS's, are adopted as additional circuit variables. For each of these branches we insert a short-circuit branch (SCB) with an intermediate node, as shown in Fig. 1, if it is not already a SCB. The currents of the SCB's are used as the circuit variables in place of the original branch currents. This process does not affect the circuit solutions, but simplifies the procedures of reduction, pivotings and formulation of the circuit matrix, as will be seen shortly, since all the branches for the current variables are then the SCB's and the VS's are distinguished from the SCB's.

The following assumptions are imposed on the resulting circuit N augmented by the SCB's:

1. All controlling currents and output currents are those through SCB's.
2. There are no loops consisting of only VS's and SCB's, and no cutsets consisting of only current sources and open-circuit branches.
3. N is connected and linearized.

Notice that only three types of elements exist in N: (i) current-defining elements such as admittances, current sources and open-circuit branches, (ii) VS's, and (iii) SCB's. Impedance type elements are converted to current-controlled VS's. Let N consist of n nodes except the reference node, b_d current-defining elements, b_v VS's, and b_s SCB's. Similarly, let N_p consist of n_p nodes except the reference node, b_{pv} VS's, and b_{ps} SCB's. Then, $n=np+(b_v-b_s)$ and $b_{pv} \leqslant b_v \leqslant b_{pv}+(b_s-b_{ps})$, since the number of nodes in N is increased by the insertion of SCB's into N_p.

According to the above assumptions, the

branch relations of the circuit elements in N are of the following form:

$$
\begin{aligned}
i_d &= H_{dd} v_d + H_{dv} v_v + H_{ds} i_s + s_d \\
v_v &= H_{vd} v_d + H_{vv} v_v + H_{vs} i_s + s_v \\
v_s &= 0
\end{aligned}
\tag{1}
$$

where i_d and v_d are the current and voltage vectors of b_d current-defining elements, i_v and v_v the current and voltage vectors of b_v VS's, i_s and v_s the current and voltage vectors of b_s SCB's, respectively. The meanings of the coefficient matrices in Eq. (1) are evident. Eq. (1) is general in the sense that it characterizes all types of elements in N (or in N_p). We partition the reduced incidence matrix A and the branch current and voltage vectors, i_b and v_b, accordingly as follows:

$$
\begin{aligned}
A &= [A_d, A_v, A_s] \\
i_b &= [i_d, i_v, i_s] \\
v_b &= [v_d, v_v, v_s]
\end{aligned}
\tag{2}
$$

Then, the MNA matrix equation for the circuit N takes the form

$$
\begin{bmatrix} A_d H_{dd} A_d^T + A_d H_{dv} A_v^T & A_v & A_s + A_d H_{ds} \\ A_v^T - H_{vd} A_d^T - H_{vv} A_v^T & 0 & -H_{vs} \\ A_s^T & 0 & 0 \end{bmatrix} \begin{bmatrix} v_n \\ i_v \\ i_s \end{bmatrix} = \begin{bmatrix} -A_d s_d \\ s_v \\ 0 \end{bmatrix}
\tag{3}
$$

where v_n is the vector of n node voltages in N and T denotes the tranpose.

Eq. (3) shows a simple form that VS currents and terminal node voltages of SCB's are related to only corresponding reduced incidence matrices, Av and A_s, respectively. In general, the current through a VS and one of the two terminal node voltages of a SCB are of no interest: these 'redundant' variables can be eliminated in Eq. (3) by simple summing operations as follows. A VS current can be eliminated by summing two row equations corresponding to the nodal equations of the VS nodes. For a grounded VS this operation results in deletion of the row equation corresponding to the nodal equation of the ungrounded node. By similar operations, the redundant node voltages associated with SCB's can be eliminated; in this case, the columns corresponding to the terminal node voltages of SCB's are summed (or deleted).

For a systematic elimination of variables, we perform two summing operations in A. One is applied to each column of A_V to get a reduced incidence matrix A_r of rank $(n-b_v)$ with $(n-b_v)$ rows, while the other to each column of A_S to get a reduced incidence matrix A_c of rank $(n-b_s)$ with (n−bs) rows. A_r and A_c can be interpreted as the reduced incidence matrices of circuits formed by collapsing terminal nodes of VS's and SCB's, respectively. With these, two types of reduced incidence matrices, we get

$$
\begin{array}{cc}
 & \begin{array}{cc} (n-b_s) & \qquad\qquad b_s \end{array} \\
\begin{array}{c} (n-b_v) \\ b_v \end{array} & \begin{bmatrix} A_{rd} H_{dd} A_{cd}^T + A_{rd} H_{dv} A_{cv}^T & A_{rs} + A_{rd} H_{ds} \\ A_{cv}^T - H_{vd} A_{cd}^T - H_{vv} A_{cv}^T & -H_{vs} \end{bmatrix} \begin{bmatrix} v_{sn} \\ i_s \end{bmatrix} = \begin{bmatrix} -A_{rd} s_d \\ s_v \end{bmatrix}
\end{array}
\tag{4}
$$

Where

$$
\begin{aligned}
A_r &= [A_{rd}, 0, A_{rs}] \\
A_c &= [A_{cd}, A_{cv}, 0] \\
v_b &= A_c^T v_{sn}
\end{aligned}
$$

Eq. (4) represents the n circuit equations of the RMNA, in which redundant variables are eliminated. The full ranks of A_r and A_s guarantee linear independence of the equations in (4). The number of variables is equal to the number of nodes in N as in the nodal approach and reduced by (b_v+b_s) as compared to Eq. (3). In terms of the original circuit N_p, the MNA requires $(n_p+b_{pv}+b_s)$ variables with $(b_{pv}+b_{ps})$ variables reduced.

We will now consider how to formulate these RMNA equations directly from the circuit N.

Remove from N the branches (leaving their terminal nodes) other than VS's and SCB's and denote the resulting subcircuit as N_{VS}. The subcircuit N_{VS} will become a subgraph of some spanning tree containing all VS's and SCB's under the assumption 2. From N_{VS}, we form two subcircuits. First, we remove all SCB's from N_{VS} and denote the resulting subcircuit as N_V. Also, we remove all VS's from N_{VS} and denote the resulting subcircuit as N_S. The subcircuit N_V consists of $(n-b_v)$ components (separate parts) excluding the component containing the reference node; some may be isolated nodes and others may contain one or several VS's with their terminal nodes. For each component C in N_V, we formulate one cutset equation encircling C (i.e., the cutset equation obtained by applying the KCL to the cutset branches of N encircling C). In total, we get $(n-b_v)$ Kirchhoff current equations, which are linearly independent since each of them is associated with a mutually exclusive node or a node set. In these equations, the contributions due to VS currents do not appear, since all VS's are contained in the components of N_V. In other words, the above cutset equations are expressed in terms of only node voltages and SCB currents; they are the cutset equations as obtained in Eq. (4). Fig. 2 shows several cutsets to be considered in the RMNA. The cutset equations corresponding to Figs 2(a), (b) and (d) will be used as the RMNA equations, while those for Fig. 2(c) and (e) will be deleted. The branch relations for VS's are b_v additional independent circuit equations, which are also expressed in terms of node voltages and SCB currents. Similarly, the subcircuit N_S consists of $(n-b_s)$ components except the component containing the reference node. Since each component in N_S contains only one node or nodes connected by SCB's, only $(n-b_s)$ node voltage variables are needed to represent all of the node voltages in N. The b_s SCB currents are used as additional circuit variables. Consequently, we use N_V for formulating the RMNA equations and N_S for adopting the RMNA variables.

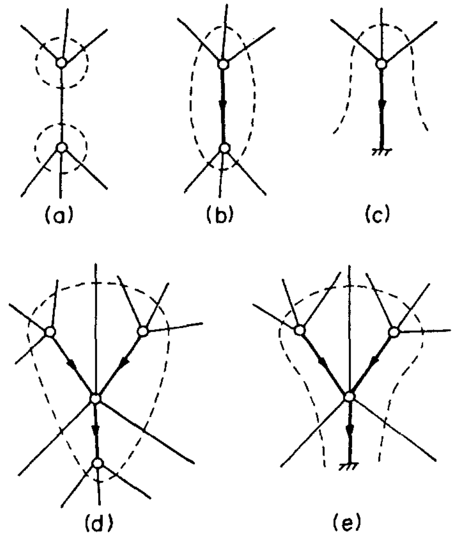


Fig. 2. RMNA cutsets. (a) Cutset encircling a node. (b) Cutset encircling a VS. (c) Cutset encircling a grounded VS. (d) Extended case of (b). (e) Extended case of (c). (Bold and dashed lines show VS's and cutsets, respectively, while the solid lines represent branches other than VS's.)

III. RMNA Pivotings

In Section II, two types of subcircuits are considered independently. However, it is possible to combine them into one subcircuit N_{VS} with an aim to avoiding zero diagonals in the matrix form of equations. Since N_V and N_S are edge-disjoint and $N_{VS}=N_V \cup N_S$, we can establish one-to-one correspondence between the set of RMNA equations and the set of RMNA variables through one node ordering process in N_{VS}.

One basic node renumbering scheme performed in N_{VS} is proposed in [5], exploiting the tree structure of N_{VS}. This scheme fits our basic aim and furthermore gives some freedom in the node ordering since, at each renumbering stage, nodes with a degree of 0 or 1 in N_{VS} may be renumbered in an arbitrary order. This freedom can be capitalized in refine our scheme further. Here, we invoke the concept of the Markowitz algorithm [6] as applied to rows only with the introduction of node weights in order to reduce the number of nonzero entries in the upper triangular part of the matrix. This simple modification can be used for the one-way node scheduling [7].

For this purpose, we propose the following strategy for sequential node renumbering to be performed in N_{VS}.

Step 1. (Short-circuit branch insertion) Construct the augmented circuit N from a given circuit N_P by inserting SCB's such that each of them satisfies the following conditions as far as possible (in the priorities listed):

(a) It is not adjacent to independent VS's:
(b) It is not adjacent to dependent VS's:
(c) It is not adjacent to SCB's:

(d) It is connected to the node with a smaller node degree in N.

Step 2. (Node renumbering procedure)
Let C be a set of candidate nodes for renumbering, D(k) a set of nodes with a degree of k in N_{VS}, and w(m) the number of unordered variables when formulating the circuit equation associated with a node m.

Initialization:
(a) Form a subcircuit N_{VS} of N.
(b) The reference node is renumbered as (n+1), the largest number.

Repeat the following steps (a)–(d) until C=0:
(a) C = D(0) U D(1) – (reference node).
(b) For each node m in C, find the node weight w(m).
(c) Renumber a node x in C, whose weight is minimum.
(d) $N_{VS} \longleftarrow N_{VS} - \{x\}$, and update D(0) and D(1).

This node renumbering procedure terminates in n repetitions. If not, one of the assumptions 2 and 3 is violated. The calculation of node weights at each iteration is straight-forward, since renumbering the node number determines the order of the corresponding equation and variable.

For the notational convenience, let P(m) be the new number of a node m after the above node renumbering procedure: MX(C) the node representing a (sub)circuit C (i.e., the node in C with the largest renumbered node number): MN(B) the node representing a circuit element B (i.e., the node of B with the smaller renumbered node number): IV(m) and IS(m) the component in N_V and N_S containing a node m, respectively; and IVS (C) the component in N_{VS} containing a component C in N_V or in N_S. With these notations, we establish one-to-one correspondence between the set of RMNA equations and the set of RMNA variables through the node renumbering in N_{VS}, as follows.

By renumbering a node m of degree 1, we can relate (i) the branch relation of its adjacent VS to the identical node voltages in IS(m), or (ii) the cutset equation encircling IV(m) to its adjacent SCB current. Also, by renumbering a node m of degree 0, we can relate (iii) the cutset equation encircling IV(m) to the identical node voltages in IS(m). We then place the cutset equation encircling IV(m) at the P(MX(IV(m)))-th row and the branch relation of a VS denoted as B at the P(MN(B))-th row. On the other hand, the voltage of a node m is corresponded to the P(MX(IS(m)))-th circuit variable, and the current of a SCB denoted as B to the P(MN(B))-th circuit variable.

From the above discussion, we define three pivoting vectors, namely, the node renumbering vector q, the row pivoting vector r and the column pivoting vector c:

$$\begin{aligned} q_j &= P(m), \\ r_j &= P(MX(IV(m))), \qquad j=1,2,\cdots,n,(n+1) \\ c_j &= P(MX(IS(m))), \end{aligned} \tag{5}$$

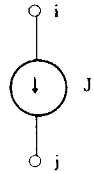
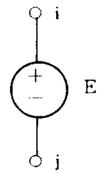
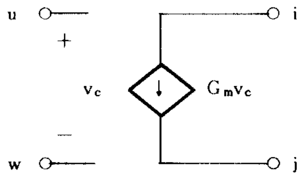
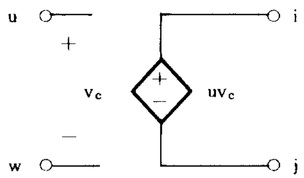
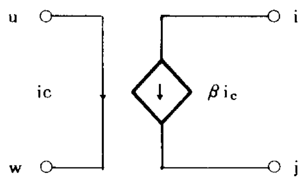
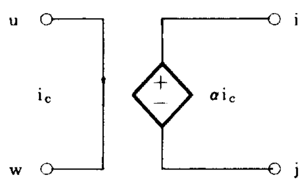
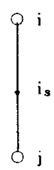
where m represents the node numbered as j in the original node numbering.

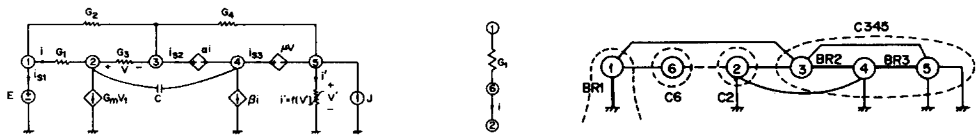
By using q and r for formulating and ordering the circuit equations, and q and c for adopting and ordering the circuit variables, the element stamps of the RMNA are easily obtained, from which the RMNA matrix equation is constructed by inspection. Table 1 shows the element stamps for several elements. Notice that each circuit element has a unique type of element stamp in the RMNA. whereas two types of element stamps may be considered in the MNA.

Example 1. Consider the circuit of Fig. 3(a), which contains two types of independent sources, four types of dependent sources, one nonlinear element, one capacitance, four conductances, and five nodes except the reference node. For the MNA, we choose three branch currents for voltage sources and one controlling current as the circuit variables, in addition to five node voltages versus the reference node. Then, the MNA matrix equation is

where $G_{eq}=\left.\frac{df}{dv}\right|_{V_5}$ $I_{eq}=f(V_5)-G_{eq}V_5$ and D denotes dt/d. According to the SCB insertion process described previously, we add a SCB to the branch of conductance of G_1, as shown in Fig. 3(b). We also apply the MNA to this modified circuit to obtain

Table 1. RMNA stamps for circuit elements.

Type	Symbol	Element stamps
Independent current source		$\begin{array}{c\|c} & \mathrm{RHS} \\ \hline r_i & -J \\ \hline r_j & J \end{array}$
Independent voltage source		$\begin{array}{c\|c\|c\|c} & c_i & c_j & \mathrm{RHS} \\ \hline \min(q_i, q_j) & 1 & -1 & E \end{array}$
VCCS		$\begin{array}{c\|c\|c\|} & c_u & c_w \\ \hline r_i & Gm & -G_m \\ \hline r_j & -G_m & G_m \end{array}$
VCVS		$\begin{array}{c\|c\|c\|c\|c} & c_i & c_j & c_u & c_w \\ \hline \min(q_i, q_j) & 1 & -1 & -u & u \end{array}$
CCCS		$\begin{array}{c\|c\|} & \min(q_u, q_w) \\ \hline r_i & \beta \\ \hline r_j & -\beta \end{array}$
CCVS		$\begin{array}{c\|c\|c\|c} & c_i & c_j & \min(q_u, q_w) \\ \hline \min(q_i, q_j) & 1 & -1 & -\alpha \end{array}$
Short-circuit branch		$\begin{array}{c\|c\|} & \min(q_i, q_j) \\ \hline r_j & 1 \\ \hline r_j & -1 \end{array}$



(a) (b) (c)

node		1	2	3	4	5	6
iteration	1	*1/0	1/4	1/2		1/3	1/2
	2		1/3	1/2		1/3	1/1
	3		0/2	1/1		1/3	
	4		0/1		1/2	1/2	
	5				1/1	1/1	
	6					1/0	
renumber		1	4	3	5	6	2
variable		v_1	$v_2=v_6$	v_3	v_4	v_5	i
equation		BR 1	C 2	BR 2	BR 3	C345	C 6

(* : node degree in N_{vs}/node weight)

(d)

Approach	MNA		RMNA	
diagonal pivotings	before	after	before	after
matrix size	9×9	9×9	6×6	6×6
# of zero diagonals	3	0	4	0
# of nonero entries	+13/33	5/33	10/22	4/22
# of fill-ins	32	7	10	1

(+ : in the upper triangular part/in the full matrix)

(e)

Fig. 3 Example 1. (a) Original circuit. (b) Modified branch of conductance of G1. (c) RMNA equations. (d) Node renumbering procedure, and equations and variables associated with nodes. (e) Comparison of the MNA and the RMNA.

$$\begin{array}{cc} & \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & is_1 & is_2 & is_3 & i \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array} & \begin{bmatrix} G_2 & 0 & -G_2 & 0 & 0 & 1 & 0 & 0 & 1 \\ G_m & G_3+CD & -G_3 & -CD & 0 & 0 & 0 & 0 & -1 \\ -G_2 & -G_3 & G_2+G_3+G_4 & 0 & -G_4 & 0 & 1 & 0 & 0 \\ 0 & -CD & 0 & CD & 0 & 0 & -1 & 1 & \beta \\ 0 & 0 & -G_4 & 0 & G_4+G_{eq} & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -\alpha \\ 0 & -\mu & \mu & 1 & -1 & 0 & 0 & 0 & 0 \\ G_1 & -G_1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ is_1 \\ is_2 \\ is_3 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ J+I_{eq} \\ E \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

$$\begin{array}{cc} & \begin{array}{cccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & is_1 & is_2 & is_3 & i \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} & \begin{bmatrix} G_1+G_2 & 0 & -G_2 & 0 & 0 & -G_1 & 1 & 0 & 0 & 0 \\ G_m & G_3+CD & -G_3 & -CD & 0 & 0 & 0 & 0 & 0 & -1 \\ -G_2 & -G_3 & G_2+G_3+G_4 & 0 & -G_4 & 0 & 0 & 1 & 0 & 0 \\ 0 & -CD & 0 & CD & 0 & 0 & 0 & -1 & 1 & \beta \\ 0 & 0 & -G_4 & 0 & G_4+G_{eq} & 0 & 0 & 0 & -1 & 0 \\ -G_1 & 0 & 0 & 0 & 0 & G_1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -\alpha \\ 0 & -\mu & \mu & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ is_1 \\ is_2 \\ is_3 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ J+I_{eq} \\ 0 \\ E \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (7)$$

For the reduction of network variables, we apply the row and column suming operations to Eq. (7) as follows:

1. Delete the first row to eliminate i_{S1}.
2. Sum the third and fourth rows to eliminate i_{S2}.
3. Sum the fourth and fith rows to eliminate i_{S3}.
4. Sum the second and sixth columns to identify v_2 with v_6.

The resulting formulation is given below.

$$
\begin{array}{r}
\\ 2 \\ 3+4+5 \\ 6 \\ 7 \\ 8 \\ 9
\end{array}
\begin{array}{c}
\begin{array}{cccccc} v_1 & v_2=v_6 & v_3 & v_4 & v_5 & i \end{array} \\
\begin{bmatrix}
G_m & G_3+CD & -G_3 & -CD & 0 & -1 \\
-G_2 & -G_3-CD & G_2+G_3 & CD & G_{eq} & \beta \\
-G_1 & G_1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & -\alpha \\
0 & -\mu & \mu & 1 & -1 & 0
\end{bmatrix}
\end{array}
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ i \end{bmatrix}
=
\begin{bmatrix} 0 \\ J+I_{eq} \\ 0 \\ E \\ 0 \\ 0 \end{bmatrix}
\tag{8}
$$

Eq. (8) shows the RMNA matrix equation which is derived from the MNA matrix equation. Eq. (8) consists of six circuit equations, but contains the zero diagonals.

The zero diagonals in Eq. (8) are avoided by the RMNA pivotings and the circuit equations of the RMNA are obtained directly from the circuit without formulating the MNA matrix equation. In Fig. 3(d) are shown the process of renumbering node numbers performed in N_{VS} the resulting q, r and c are obtained as

$$
\begin{aligned}
q &= [1, 4, 3, 5, 6, 2, 7] \\
r &= [7, 4, 6, 6, 6, 2, 7] \\
c &= [1, 4, 3, 5, 6, 4, 7]
\end{aligned}
$$

Therefore, we have the following RMNA matrix equation:

$$
\begin{array}{r}
\\ BR1 \\ C6 \\ BR2 \\ C2 \\ BR3 \\ C345
\end{array}
\begin{array}{c}
\begin{array}{cccccc} v_1 & i & v_3 & v_2=v_6 & v_4 & v_5 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-G_1 & 1 & 0 & G_1 & 0 & 0 \\
0 & -\alpha & 1 & 0 & -1 & 0 \\
G_m & -1 & -G_3 & G_3+CD & -CD & 0 \\
0 & 0 & \mu & -\mu & 1 & -1 \\
-G_2 & \beta & G_2+G_3 & -G_3-CD & CD & G_{eq}
\end{bmatrix}
\end{array}
\begin{bmatrix} v_1 \\ i \\ v_3 \\ v_2 \\ v_4 \\ v_5 \end{bmatrix}
=
\begin{bmatrix} E \\ 0 \\ 0 \\ 0 \\ 0 \\ J-I_{eq} \end{bmatrix}
\tag{9}
$$

Note that, as we intended, Eq. (9) consists of six circuit equations which are equivalent to those in Eq. (8), and its matrix has no zero diagonals and is sparse especially in the upper triangular matrix, which tends to reduce the number of fill-ins generated in the subsequent LU decomposition process. For this particular example, comparisons between the MNA and the RMNA are made as given in Fig. 3(e). Here, the row-interchange in [1] and the Markowitz algorithm as applied to rows only are also applied to Eq. (6) for fair comparison. The RMNA pivotings automatically avoid zero diagonals and, as we expect, reduce the number of fill-ins as well as the number of nonzero entries and the matrix size, as compared with the MNA. Another interesting fact is that contribution of conductance G_4 does not appear in Eq. (9). In general, any element parallel with a VS or a series combination of VS's does not affect the circuit solutions, and hence it is excluded in the RMNA matrix form.

IV. Difference from Other MNA Methods

The two-graph modified nodal formulation in [4] intends to eliminate redundant circuit variables by the use of separate voltage and current graphs, which resembles the subcircuits N_V and N_S, respectively, in our approach. However, the voltage and current graphs may differ not only in structure but also in the numbers of nodes and edges, and hence they cannot be combined into one graph for the purpose of one node renumbering process, which makes it difficult to perform pivotings at the formulation stage to avoid zero diagonals and, at the same time, to reduce the number of fill-ins. In contrast, the RMNA uses one subcircuit $N_{VS}=N_V \cup N_S$, which is possible because N_V and N_S contain the same nodes and are edge disjoint due to the inserted SCB's. The pivoting procedure in [2] also aims at avoiding zero diagonals in the MNA matrix, but it does not consider reduction of the circuit variables. These points are illustrated in Examples 3 and 4, respectively.

Example 3. In order to compare the two-graph modified nodal formulation with the RMNA, we apply the RMNA to the circuit of Example 4.8.1 in [8], which is reproduced in Fig. 6. Then,

$$
\begin{aligned}
q &= [1,3,4,2,5] \\
r &= [5,3,4,5,5] \\
c &= [1,3,4,2,5]
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\mu \\ -G_2 & -sC_4 & G_2+G_3+sC_4 & -G_3 \\ 0 & 0 & -G_3 & G_3+sC_5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_4 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}
$$

The network equations in Eq. (12) are the same as those in Example 4.8.1 of [4], but they are already so ordered as to avoid zero diagonals. Furthermore, the matrix in Eq. (10) generates no fill-ins during the subsequent LU decomposition process.

Example 4. In order to compare the pivoting method in [2] with the RMNA, we apply the RMNA to the circuit of Example 3 in [2], which is reproduced in Fig. 7 except that a short-circuit branch is inserted in series with the independent voltage source E for the controlling current i_1. Then,

$$
\begin{aligned}
q &= [2,1,3,4,5] \\
r &= [4,4,3,4,5] \\
c &= [2,1,3,5,5]
\end{aligned}
$$

$$
\begin{bmatrix} 1 & -1 & -\beta & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 3 & 0 \\ 2 & 1 & -2 & 1-\alpha \end{bmatrix} \begin{bmatrix} v_2 \\ v_1 \\ i_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ E \\ 0 \\ 0 \end{bmatrix} \tag{11}
$$

As we expect, the current variable i_2 does not show up in (13). As a result, the number of variables is reduced by 1 as well as the number of nonzero entries is reduced by 4, as compared with Example 3 in [2].

V. Zero Diagonal Problem

In order to treat the zero diagonal problem in the RMNA matrix, we first state the following four lemmas.

Lemma 1. In the RMNA matrix, the rows associated with branch relations have topologically nonzero diagonals, '1' or '−1'.

Proof: For a VS denoted as B, consider those nodes which are still connected with MN(B) in N_{VS} after removing B. Then, MN(B) is renumbered as the largest number among those nodes, which implies that MN(B)=MX (IS(MN(B))). This completes the proof.

Lemma 2. In the RMNA matrix, the columns associated with SCB currents have topologically nonzero diagonals, '1' or '−1'.

Proof: Similar to that of Lemma 1.

Lemma 3. In the RMNA matrix, the rows associated with cutset equations have nonzero diagonals, if and only if each ungrounded component C in N_V, for which the condition MX(C)=MX(IVS(C)) holds, has at least one cutset branch whose current is controlled by the node voltage of MX(C).

Proof: Consider an ungrounded component C in N_V. If MX(C)=MX(IVS(C)), the branch stated in the Lemma contributes to the P(MX(C))-th diagonal entry: otherwise, MX(C) is always incident with a SCB which is represented by MX(C). This completes the proof.

Lemma 4. If the circuit includes no coupled-elements and has n_d $(=n-b_v-b_s)$ admittances, which form a spanning tree including all VS's and SCB's and each of which is incident with at least one of the nodes representing the components of N_{VS}, then the leading principal minors of the RMNA matrix for any diagonal pivoting are all nonzero.

Proof: First, we derive the RMNA matrix by partitioning the node voltages. In the RMNA, each of VS's and SCB's is related to one of its two terminal nodes. Let v_{nv} and v_{ns} be the sets of node voltages associated with b_v VS's and b_s SCB's, respectively, and v_{nd} the set of remaining n_d node voltages. With this partitioning, we have the following matrix equation of the RMNA:

$$
\begin{bmatrix} A_{cvv}^T & -H_{vs} & A_{cdv}^T \\ A_{rsd}H_{dd}A_{cvd}^T & A_{rss} & A_{rsd}H_{dd}A_{cdd}^T \\ A_{rdd}H_{dd}A_{cvd}^T & A_{rds} & A_{rdd}H_{dd}A_{cdd}^T \end{bmatrix} \begin{bmatrix} v_{nv} \\ i_s \\ v_{nd} \end{bmatrix} = \begin{bmatrix} s_v \\ -A_{rsd}s_d \\ -A_{rdd}s_d \end{bmatrix} \tag{12}
$$

where

$$
A = [A_d,\ A_v,\ A_s] = \begin{array}{c} \begin{array}{ccc} b_d & b_v & b_s \end{array} \\ \left[\begin{array}{ccc} A_{dd} & A_{dv} & A_{ds} \\ A_{vd} & A_{vv} & A_{vs} \\ A_{sd} & A_{sv} & A_{ss} \end{array}\right] \end{array} \begin{array}{c} \\ n_d \\ b_v \\ b_s \end{array}
$$

$$
A_r = [A_{rd},\ 0,\ A_{rs}] = \left[\begin{array}{ccc} A_{rdd} & 0 & A_{rds} \\ A_{rsd} & 0 & A_{rss} \end{array}\right] \begin{array}{c} n_d \\ b_s \end{array}
$$

$$
A_c = [A_{cd},\ A_{cv},\ 0] = \left[\begin{array}{ccc} A_{cdd} & A_{cdv} & 0 \\ A_{cvd} & A_{cvv} & 0 \end{array}\right] \begin{array}{c} n_d \\ b_v \end{array}
$$

Comments. (a) Under the assumption of no coupled-elements, H_{vv}, H_{vd}, H_{dv}, and H_{ds} in (1) become zero coefficient matrices and H_{dd} is a diagonal matrix with positive nonzero entries. (b) Eq. (12) represents the same equations as those in Eq. (4), but interchange of rows and columns is performed so as to avoid zero diagonal entries. (c) By a proper diagonal pivoting, Eq. (12) becomes identical to the result obtained by the RMNA pivotings. (d) A_{vv}, A_{ss}, A_{rss} and A_{cvv} are square and nonsingular with nonzero diagonal entries, since all VS's and SCB's form a subgraph of some spanning tree. (e) The first $(b_v + b_s)$ diagonals in the matrix of (10) are 1 or -1, as stated in Lemmas 1 and 2. (f) A_{cvv} and A_{rss} have all nonzero leading principal minors for any diagonal pivoting. (g) The columns in A_{dd} corresponding to the n_d elements stated in the Lemma are linearly independent. And this independence is maintained also in A_{rdd} and A_{cdd}. (h) A_{dd}, A_{rdd} and A_{cdd} are n_d by b_d matrices with rank n_d. (i) $A_{rdd}\ H_{dd}\ A_{cdd}$ is nonsingular with nonzero diagonal entries, as stated in Lemma 3. (j) $A_{rdd}\ H_{dd}\ A_{cdd}$ has all nonzero leading principal minors for any diagonal pivoting.

We will now show that the matrix in Eq. (12) has nonzero leading principal minors for any diagonal pivoting. Let A(ab) denote a submatrix of a given matrix A formed by the intersection of the rows and columns of A corresponding to integer sets a and b, respectively, and also A(a.) a submatrix formed by the rows of A corresponding to the integer set a. Consider any principal submatrix of the matrix in Eq. (12) by choosing suitable index sets a, b, and c. The determinant of this submatrix can be written as which is nonzero. In Eq. (13) $B_{rsd} = A^{-1}_{rss}\ A_{rsd}$, $B_{rdd} = A_{rdd} - A_{rds}\ A^{-1}_{rss}\ A_{rsd}$, $B_{cvd} = A^{-1}_{cvv} A_{cvd}$, $B_{cdd} = A_{cdd} - A_{cdv} A^{-1}_{cvv} A_{cvd}$. Note that Brdd is equal to B_{cdd}, since they can be interpreted as the reduced incidence matrix of the circuit formed by collapsing all the terminal nodes of VS's and SCB's.

From the above discussion, it is evident that the matrix in Eq. (12) has all nonzero leading principal minors for any diagonal pivoting. This completes the proof.

$$
\begin{aligned}
&\det\left(\left[\begin{array}{ccc} A^T_{cvv}(aa) & -H_{vs}(ab) & A^T_{cdv}(ac) \\ A_{rsd}(b.)\,H_{dd}A^T_{cvd}(a.) & A_{rss}(bb) & A_{rsd}(b.)\,H_{dd}A^T_{cdd}(c.) \\ A_{rdd}(c.)\,H_{dd}A^T_{cvd}(a.) & A_{rds}(cb) & A_{rdd}(c.)\,H_{dd}A^T_{cdd}(c.) \end{array}\right]\right) \\
&= \det\left(A^T_{cvv}(aa)\right)\ \det\left(\left[\begin{array}{cc} A_{rss}(bb) & 0 \\ A_{rds}(cb) & I \end{array}\right]\left[\begin{array}{cc} I & B_{rsd}(b.) \\ 0 & B_{rdd}(c.) \end{array}\right]\left[\begin{array}{cc} I & 0 \\ 0 & H_{dd} \end{array}\right]\right. \\
&\qquad \left.\left[\begin{array}{cc} I & 0 \\ B^T_{cvd}(a.)\,H_{vs}(ab) & B^T_{cdd}(c.) \end{array}\right]\right) \qquad (13)
\end{aligned}
$$

The above four lemmas show that the RMNA avoids topologically zero diagonals in its matrix form. Especially, Lemma 4 guarantees that the LU decomposition process of the RMNA matrix will not fail for any diagonal pivoting. The condition of Lemma 4 requires existence of n_d admittances which connect all the components in N_{vs} and satisfy the condition of Lemma 3. This condition is always satisfied, when the circuit contains a spanning tree of only admittances which are not parallel with a VS or a SCB or their series combination – a probable situation in most practical circuits. Note that any element parallel with a VS or a series combination of VS's does not contribute to the RMNA equations. Furthermore, this condition automatically holds in all cases, if we use a spanning tree of admittances including all VS's and SCB's instead of the subcircuit N_{vs} at the node renumbering stage and renumber nodes with a degree of 1, sequentially. The following example clearifies this point.

Example 2. For the circuit given in Fig. 4, we obtain two cases of node ordering, each of which obey both the 'positive node selection algorithm' given in Appendix of Ref. [2] and the RMNA pivotings in our approach. For each case, the MNA matrix and the RMNA matrix are obtained.

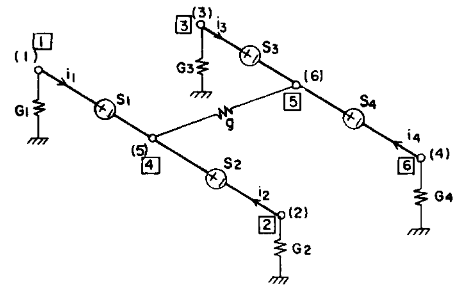


Fig. 4. Circuit of Example 2. Case 1 and Case 2 are represented by the numbers in the parenthesis and the rectangle, respectively.

(Case 1)

Reduced incidence matrix

$$\begin{array}{cc} & \begin{array}{ccccccccc} S1 & S2 & S3 & S4 & G1 & G2 & G3 & G4 & g \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \left[\begin{array}{ccccccccc} 1 & & & & 1 & & & & \\ & 1 & & & & 1 & & & \\ & & 1 & & & & 1 & & \\ & & & 1 & & & & 1 & \\ -1 & -1 & & & 0 & 0 & 0 & 0 & 1 \\ & & -1 & -1 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \end{array}$$

MNA matrix

$$\begin{array}{c} \begin{array}{cccccccccc} i_1 & i_2 & i_3 & i_4 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \left[\begin{array}{cccccccccc} 1 & & & & G1 & & & & & \\ & 1 & & & & G2 & & & & \\ & & 1 & & & & G3 & & & \\ & & & 1 & & & & G4 & & \\ & & & & 1 & & & & -1 & \\ & & & & & 1 & & & -1 & \\ & & & & & & 1 & & & -1 \\ & & & & & & & 1 & & -1 \\ -1 & -1 & & & & & & & g & -g \\ & & -1 & -1 & & & & & -g & g \end{array}\right] \end{array}$$

RMNA matrix

$$\begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & & & & -1 & \\ & 1 & & & -1 & \\ & & 1 & & & -1 \\ & & & 1 & & -1 \\ G1 & G2 & & & g & -g \\ & & G3 & G4 & -g & g \end{array}\right] \end{array}$$

(Case 2)

Reduced incidence matrix

$$\begin{array}{cc} & \begin{array}{ccccccccc} S1 & S2 & S3 & S4 & G1 & G2 & G3 & G4 & g \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \left[\begin{array}{ccccccccc} 1 & & & & 1 & & & & \\ & 1 & & & & 1 & & & \\ & & 1 & & & & 1 & & \\ & & -1 & -1 & & & & & -1 \\ -1 & -1 & & & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

MNA matrix

$$\begin{array}{c} \begin{array}{cccccccccc} i_1 & i_2 & i_3 & i_4 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \left[\begin{array}{cccccccccc} 1 & & & & G1 & & & & & \\ & 1 & & & & G2 & & & & \\ & & 1 & & & & G3 & & & \\ & & -1 & -1 & & & & g & -g & \\ & & & & 1 & & & & -1 & \\ & & & & & 1 & & & -1 & \\ & & & & & & 1 & -1 & & \\ & & & & & & & -1 & & 1 \\ -1 & -1 & & & & & & -g & g & \\ & & & 1 & & & & & & G4 \end{array}\right] \end{array}$$

RMNA matrix

$$\begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \left[\begin{array}{cccccc} 1 & & & & -1 & \\ & 1 & & & -1 & \\ & & 1 & -1 & & \\ & & & -1 & 1 & \\ G1 & G2 & & -g & g & \\ & & G3 & g & -g & G4 \end{array}\right] \end{array}$$

In Case 1, both the MNA and the RMNA matrix have a singular submatrix. This singularity arises from that the submatrix A_{dd} of the reduced incedence matrix is not of full rank. Hence, the positive node selection algorithm and the RMNA node renumbering fail to select the admittances which satisfy the condition of Lemma 4. That is, the admittances do not form a spanning tree of the circuit obtained by grounding all the nodes representing VS's and SCB's and removing all VS's and SCB's. In contrast, Case 2 shows that both the MNA and the RMNA matrix have no nonsingular principal minors and the submatrix A_{dd} is of full rank.

Although the above four lemmas guarantees the nonzero diagonals in the RMNA matrix, zero diagonals may still occur in very special element connections with specific element values and types. Fig. 5 shows the only cases of zero diagonals occurring in treating VS's and SCB's, regardless of Lemmas 1 and 2. A simple case which does not satisfy the condition of Lemma 3, is seen when f(v) is removed from Fig. 3(a) in Example 1; then the sixth diagonal entry in Eq. (9) becomes zero. Except for these three cases, the RMNA pivotings always generate nonzero diagonals. If zero diagonals appear in the RMNA matrix – a very rare case in practical circuits, they do so in the nodal matrix and in the MNA matrix.

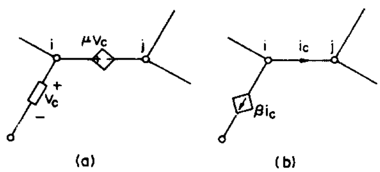


Fig. 5. Occurrence of zero diagonals. (a) A voltage-controlled voltage source is adjacent to its controlling branch, and $\mu = 1$ and $q_i < q_j$. (b) A current-controlled current source is adjacent to its controlling branch, and $\beta = -1$ and $q_i < q_j$.

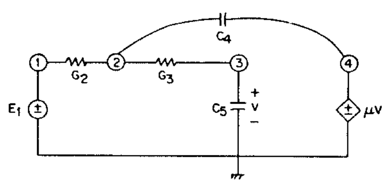


Fig. 6. Circuit of Example 3.

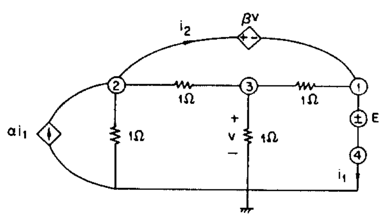


Fig. 7. Circuit of Example 4.

VI. Conclusions

In this paper, a new and general nodal formulation for circuit analysis has been proposed, in which a special family of cutset equations as well as the branch relations for VS's are formulated in terms of node voltages and controlling currents. This approach is based on the MNA, but treats effectively VS's and SCB's to eliminate redundant variables. In the RMNA, the number of circuit variables is always equal to that of nodes in the circuit augmented by the SCB's. Furthermore, the RMNA pivotings enable the direct formulation of the circuit equations by inspection and avoid zero diagonal entries in its matrix, which is sparse in the upper triangular part. Also, they can be combined with other pivoting processes utilizing some freedom in the node ordering, as stated in Section III. Experimental results for various circuits show that the RMNA pivotings tend to reduce the number of fill-ins in the subsequent LU decomposition as well as the number of nonzero entries, as compared with the row-interchange used in the MNA.

Incidentally, the RMNA pivoting schemes and the SCB insertion process do not require severe computational overhead and programming effort, since in most CAD programs it is necessary anyway to test the circuit topology against the existence of a loop consisting of solely VS's and SCB's, to examine the element type for adopting the current variables, and to go through pivotings, for avoiding zero diagonals and minimizing the number of fill-ins; our pivoting procedure can be effectively carried out during this process. Once the pivoting vectors are obtained for a circuit to be analyzed, they are used through the whole analysis procedure without any changes.

References

[1] C.W. Ho, A.E. Ruehli, and P.A. Brennan, "The modified nodal approach to network analysis," *IEEE Trans. Circuits Systs.*, vol. CAS-22, pp. 504-509, June 1975.

[2] I.N. Hajj, P. Yang and T.N. Trick, "Avoiding zero pivots in the modified nodal approach," *IEEE Trans. Circuits Systs.*, vol. CAS-28, pp. 271-279, April 1981.

[3] S.C. Fang and Y.P. Tsividis, "Modified nodal analysis with improved numerical methods for switched capacitive networks," *IEEE Proc. of ISCAS*, Houston Texas USA, pp. 977-980, April 1980.

[4] J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design*, New York: Van Nostrand Reinhold Co., New York, pp. 132-140, 1983.

[5] Kijun Lee and S.B. Park, "The reduced modified nodal approach to network analysis," *IEEE Proc. of ISCAS*, Newport Beach, California, pp. 637-640, May 1983.

[6] H.M. Markowitz, "The elimination form of the inverse and its application to linear programming," *Management Sci.*, vol. 3, pp. 255-269, 1957.

[7] A.E. Ruehli and G.S. Ditlow, "Circuit analysis, logic simulation, and design verification for VLSI," *Proc. of the IEEE*, vol. 71, no. 1, Jan. 1983.