Design of a Guaranteed Cost Controller for a Class of Systems with Uncertain Parameters

(不確定 파라미터를 갖는 시스템에 대한 Guaranteed Cost 制御機의 設計)

李 廷 紋*, 崔 桂 根*

(Jung Moon Lee and Keh Kun Choi)

要 約

本 論文에서는 不確定 파라미터를 갖는 시스템에 대한 guaranteed cost 制御機의 設計 方法을 다루었다. 設計 過程은 어떤 조건을 만족하는 임의의 함수를 定義하고 그것을 制御 入力에 관하여 最小化함으로써 이루어진다.

이것을 不確定 파라미터를 갖는 선형 시스템에 적용하면 몇 개의 항이 추가된 Riccati 方程式이 유도된다. 또한 간단한 예제를 통하여 이 方法의 有用性을 檢討해 보았다.

Abstract

This paper describes a method to design a guaranteed cost controller for a system with uncertain parameters. The design procedure consists of defining an arbitrary function which satisfies certain condition and minimizing it over the control input. When the method is applied to a class of linear systems with uncertain parameters, a Riccati equation with additional terms results. A simple example is presented to illustrate the usefulness of this method.

I. Introduction

A control problem is to determine a control which minimizes a given cost functional and satisfies all state and control constraints. If, however, there are uncertainties in the system parameters, the minimization of the cost functional over the control input cannot be carried out normally. Consequently, the design of a controller for a system with these uncertain parameters requires the solution to a modified problem.

*正會員, 서울大學校 電子工學科

(Dept. of Elec. Eng., Seoul National Univ.)

接受日字 : 1986年 2月 18日

Numerous design methods for this type of controllers have already been proposed. The minmax approach [1] gives the minimum guaranteed cost but often requires complicated computation. The sensitivity approach [2] is applicable to the systems with small parameter uncertainty. The stability approach [3-8] is concerned with only system stability.

This paper describes a method to design a guaranteed cost controller for a system with uncertain parameters which are not completely determined by measurement or vary within given bounds. Design procedure consists of defining an arbitrary function which satisfies certain condition and minimizing it over the control input.

Applying it to a class of linear systems with uncertain parameters yields a Riccati equation with additional terms which are easy and free to construct.

Finally, a simple example is presented to illustrate the usefulness of this method.

II. General Approach

The system with uncertain parameters can be described by

$$\dot{x}(t) = f(x(t), u(t), q(t), t) \quad (1)$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, and $q(t) \in R^r$ is the vector of uncertain parameters constrained to lie within the compact region Ω.

The cost functional is given as

$$J = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), q(t), t)\,dt \quad (2)$$

where $h(x(t_f), t_f)$ represents the terminal cost and the integral represents accumulated cost along the path.

The problem is to design a state feedback control law

$$u(t) = v(x(t), t) \quad (3)$$

such that the cost J does not exceed a number V for any allowed variation of q(t). In this case, V is called a guaranteed cost for the system starting from $x(t_0)$ at time t_0, and $v(x(t), t)$, $t \in [t_0, t_f]$ is called a guaranteed cost control.

Lemma II.1

Let $F(V(x(t), t), x(t), u(t), t)$ be an arbitrary function satisfying

$$F(V(x(t),t), x(t), u(t), t) \geqslant g(x(t), u(t), q(t), t) + \frac{\partial V(x(t),t)}{\partial x} f(x(t), u(t), q(t), t) + \frac{\partial V(x(t),t)}{\partial t} \quad (4)$$

for all $q(t) \in \Omega$, x(t), u(t), and $t \in [t_0, t_f]$, where V(x(t),t) is a scalar function with continuous first partial derivatives. Let v(x(t),t) denote the value of u(t) that minimizes F(V(x(t),t), x(t),u(t),t).

Then $V(x(t_0), t_0)$ is a guaranteed cost, and $v(x(t),t)$, $t \in [t_0, t_f]$ is a guaranteed cost control for the problem defined by (1) and (2), if the following equalities are satisfied;

$$F(V(x(t),t), x(t), v(x(t),t), t) = 0 \quad \text{for all } x(t) \text{ and } t \in [t_0, t_f] \quad (5.a)$$

$$V(x(t_f), t_f) = h(x(t_f), t_f) \quad (5.b)$$

Proof: Combining (4) and (5.a) gives

$$g(x(t), v(x(t),t), q(t), t) + \frac{\partial V(x(t),t)}{\partial x} f(x(t), v(x(t),t), q(t), t) + \frac{\partial V(x(t),t)}{\partial t} \leqslant 0 \quad (6)$$

for all $q(t) \in \Omega$, x(t), and $t \in [t_0, t_f]$. The condition (6) can be written as

$$-\frac{dV(x(t), t)}{dt} \geqslant g(x(t), v(x(t), t), q(t), t) \quad (7)$$

Integrating (7) with respect to $t \in [t_0, t_f]$ gives

$$V(x(t_0), t_0) - V(x(t_f), t_f) \geqslant \int_{t_0}^{t_f} g(x(t), v(x(t),t), q(t), t)\,dt \quad (8)$$

Combining (5.b) and (8) and rearranging terms give

$$V(x(t_0), t_0) \geqslant h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), v(x(t),t), q(t), t)\,dt \quad (9)$$

Hence, $V(x(t_0), t_0)$ is a guaranteed cost, and $v(x(t),t)$, $t \in [t_0, t_f]$ is a guaranteed cost control.

■

Remark II.1: The guaranteed cost control v(x(t),t) proposed by Lemma II.1 is not unique for a given guaranteed cost $V(x(t_0), t_0)$, because that Lemma II.1 is only a sufficient condition and that the function F(V(x(t),t), x(t), u(t),t) can be defined almost arbitrarily.

Remark II.2: The stability analysis for this type of problem is made by Chang and Peng[9] in the sense of Lyapunov[10].

III. Application to a Class of Linear Systems with Uncertain Parameters

1. System Description

The linear system under consideration is described by

$$\dot{x}(t) = [A_0 + \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} r_{ij}(t)]\, x(t) + [B_0 + \sum_{i=1}^{n}\sum_{j=1}^{m} B_{ij} s_{ij}(t)]\, u(t) \quad (10)$$

where the uncertain parameters are constrained by

$$|r_{ij}(t)| \leqslant 1 \quad i = 1,2, \dots, n; j = 1,2,\dots, n \quad (11.a)$$
$$|s_{ij}(t)| \leqslant 1 \quad i = 1,2,\dots, n; j = 1,2,\dots, m \quad (11.b)$$

for all $t \in [t_0, t_f]$. The matrices A_{ij} and B_{ij} are assumed to be of the form

$$A_{ij} = d_{ij} e'_{ij} \quad i = 1,2, \dots, n; j = 1,2, \dots, n \quad (12.a)$$
$$B_{ij} = f_{ij} g'_{ij} \quad i = 1,2, \dots, n; j = 1,2, \dots, m \quad (12.b)$$

where $d_{ij} \in R^n$ and $f_{ij} \in R^n$ have 0's in all but the i-th positions, and $e_{ij} \in R^n$ and $g_{ij} \in R^m$ have 0's in all but the j-th positions.

Remark III.1: The decompositions required by (12.a) and (12.b) are not unique. This fact may be utilized by the designer.

In the sequel, we will use the following definitions;

$$D \triangleq \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} d'_{ij} \quad (13.a)$$

$$E \triangleq \sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij} e'_{ij} \quad (13.b)$$

$$F \triangleq \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij} f'_{ij} \quad (13.c)$$

$$G \triangleq \sum_{i=1}^{m}\sum_{j=1}^{m} g_{ij} g'_{ij} \quad (13.d)$$

The matrices defined above are all diagonal.

The cost functional is given as

$$J = \frac{1}{2} x'(t_f) H x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x'(t) Q x(t) + u'(t) R u(t)]\, dt \quad (14)$$

where H and Q are symmetric, positive semidefinite matrices, and R is a symmetric, positive definite matrix.

2. Design Procedure

We will find a guaranteed cost solution of the form

$$V(x(t),t) = \frac{1}{2} x'(t) K(t) x(t) \quad (15)$$

where K(t) is a symmetric, positive definite matrix for all $t \in (t_0, t_f)$.

Lemma III.1

Let F(V(x(t),t),x(t), u(t),t) be defined as

$$\begin{aligned} F(V(x(t),t),x(t),u(t),t) &\triangleq \frac{1}{2} [x'(t) Q x(t) + u'(t) R u(t)] \\ &+ x'(t) K(t)\, [A_0 x(t) + B_0 u(t)] \\ &+ \frac{1}{2} [x'(t) K(t) D K(t) x(t) + x'(t) E x(t) \\ &+ x'(t) K(t) F K(t) x(t) + u'(t) G u(t)] \\ &+ \frac{1}{2} x'(t) \dot{K}(t) x(t) \end{aligned} \quad (16)$$

Then the condition (4) is satisfied.

Proof: The right-hand side of (4) becomes

$$\begin{aligned} & g(x(t),u(t),q(t),t) + \frac{\partial V(x(t),t)}{\partial x} f(x(t),u(t),q(t),t) + \frac{\partial V(x(t),t)}{\partial t} \\ &= \frac{1}{2}[x'(t) Q x(t) + u'(t) R u(t)] + x'(t) K(t) \\ &([A_0 + \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} r_{ij}(t)] x(t) + [B_0 + \sum_{i=1}^{n}\sum_{j=1}^{m} B_{ij} s_{ij}(t)] u(t) \\ &+ \frac{1}{2} x'(t) \dot{K}(t) x(t) \end{aligned} \quad (17)$$

In order to obtain the upper bound of the terms containing uncertain parameters in (17), we will use the following inequality [11];

$$
\begin{aligned}
& x'(t)K(t) \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\, r_{ij}(t)x(t) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} x'(t)K(t)d_{ij}e'_{ij}x(t)r_{ij}(t) \\
&\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} |x'(t)K(t)d_{ij}e'_{ij}x(t)| \\
&\leqslant \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x'(t)K(t)d_{ij}d'_{ij}K(t)x(t) \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x'(t)e_{ij}e'_{ij}x(t) \\
&= \frac{1}{2}x'(t)K(t)DK(t)x(t) + \frac{1}{2}x'(t)Ex(t) \qquad (18)
\end{aligned}
$$

One can also derive a similar inequality for the remaining term. These inequalities prove the lemma. ■

Based on Lemma II.1, a guaranteed cost control $v(x(t),t)$ can be found as

$$u(t) = v(x(t),t) = -(R + G)^{-1}B_0'K(t)x(t) \quad (19)$$

by differentiating (16) with respect to $u(t)$. Substituting (19) into (16) and rearranging terms give

$$
\begin{aligned}
& F(V(x(t),t),x(t),v(x(t),t),t) \\
&= \frac{1}{2} x'(t)(K(t)A_0 + A_0'K(t) + Q + E \\
&\quad - K(t)\,[B_0(R + G)^{-1}B_0' - (D + F)]\,K(t) \\
&\quad + \dot{K}(t)\,)\, x(t) \qquad (20)
\end{aligned}
$$

Then the condition (5.a) and (5.b) are implied by the following;

$$
\begin{aligned}
-\dot{K}(t) &= K(t)A_0 + A_0'K(t) + Q + E \\
&\quad -K(t)\,[B_0(R + G)^{-1}B_0' - (D+F)]\,K(t) \qquad (21.a)
\end{aligned}
$$

$$K(t_f) = H \qquad (21.b)$$

Remark III.2: If the matrices D and F are chosen so that $B_0(R+G)^{-1}B_0'-(D+F)$ remains positive semidefinite to ensure the positive definite solution of the Riccati equation (21.a) for all $t\in[t_0,t_f)$, the feedback control (19) guarantees that the cost (14) will not become larger than $\frac{1}{2}\, x'(t_0)K(t_0)x(t_0)$ for any allowed variation of uncertain parameters.

IV. Illustrative Example

Let us consider the system and the cost functional given by (10) and (14) respectively, where

$$
A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

$$
H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 1 \qquad (22)
$$

and $|r_{22}(t)| \leqslant 1$ for all $t \in [0,15]$. The initial state is given by

$$x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad (23)$$

If the matrix A_{22} is decomposed as

$$A_{22} = d_{22}e_{22}' = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} \qquad (24)$$

the Riccati equation (21.a) yields the solution as shown in Fig. 1.

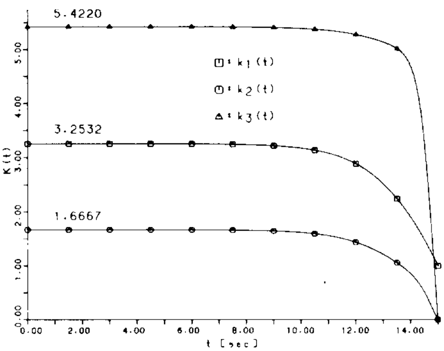

Fig. 1. Solution to the Riccati equation (21.a);

$$k(t) = \begin{bmatrix} k_1(t) & k_2(t) \\ k_2(t) & k_3(t) \end{bmatrix}.$$

The cost yielded by the control (19) is shown in Fig. 2 as a function of parameter uncertainty r_{22} which is assumed to be constant but unknown.

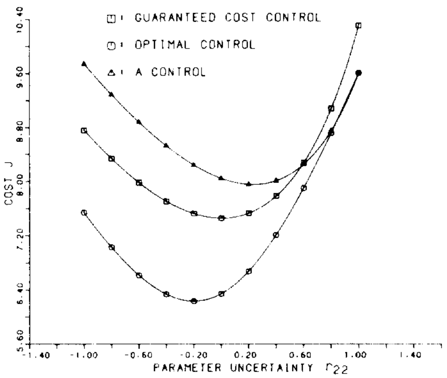

Fig. 2. Calculated cost as a function of constant parameter uncertainty r_{22}.

We observe that the cost does not exceed the guaranteed cost

$$\frac{1}{2}x'(0)K(0)x(0) = 12.5508 \tag{25}$$

for all $r_{22} \in [-1,1]$.

For comparison, it is possible to determine the optimal control for each fixed choice of r_{22}, which gives the curve marked by circles in Fig. 2. Furthermore, it should be noted that the guaranteed cost control (19) is not unique for the guaranteed cost given by (25). For example, a constant feedback gain control

$$u(t) = -1.16x_1(t) - 5.07x_2(t) \tag{26}$$

also yields a cost less than 12.5508 as is illustrated by the curve marked by triangles in Fig. 2.

V. Conclusion

This paper described an approach to determine a guaranteed cost control satisfying a sufficient condition. Current method is more flexible than the previous one [9] in the sense that a part of the procedure is almost at the designer's disposal.

Specifically, applying this method to a class of linear systems with uncertain parameters results in a Riccati equation with additional terms which are easy and free to construct.

A simple example was presented to illustrate design procedure, which produced a guaranteed cost controller for a system with large parameter variation.

References

[1] W.E. Schmitendorf, "Minmax control of systems with uncertainty in the initial state and in the state equations," *IEEE Trans. Automat. Contr.*, vol. AC-22, no. 3, pp. 439-443, June 1977.

[2] M. Sobral, Jr., "Sensitivity in optimal control systems," *Proc. IEEE,* vol. 56, no. 10, pp. 1644-1652, Oct. 1968.

[3] J.S. Thorp and B.R. Barmish, "On guaranteed stability of uncertain linear systems via linear control," *J. Optimiz. Theory Appl.,* vol. 35, no. 4, pp. 559-579, Dec. 1981.

[4] B.R. Barmish, "Necessary and sufficient conditions for quadratic stabilizability of an uncertain system," *J. Optimiz. Theory* Appl., vol. 46, no. 4, pp. 399-408, Aug. 1985.

[5] J.L. Willems and J.C. Willems, "Robust stabilization of uncertain systems," *SIAM J. Contr. Optimiz.*, vol. 21, no. 3, pp. 352-374, May 1983.

[6] E. Noldus, "Design of robust state feedback laws," *Int. J. Contr.*, vol. 35, no. 6, pp. 935-944, 1982.

[7] B.R. Barmish, M. Corless, and G. Leitmann, "A new class of stabilizing controllers for uncertain dynamical systems," *SIAM J. Contr. Optimiz.*, vol. 21, no. 2, pp.246-255, Mar. 1983.

[8] I.R. Petersen, "Structural stabilization of uncertain systems: necessity of the matching condition," *SIAM J. Contr. Optimiz.,* vol. 23, no. 2, pp. 286-296, Mar. 1985.

[9] S.S.L. Chang and T.K.C. Peng, "Adaptive guaranteed cost control of systems with uncertain parameters," *IEEE Trans. Automat. Contr.*, vol. AC-17, no. 4, pp. 474-483, Aug. 1972.

[10] J.J. D'Azzo and C.H. Houpis, *Linear Control System Analysis and Design,* McGraw-Hill, New York, pp. 485-522, 1981.

[11] I.R. Petersen, "A Riccati equation approach to the design of stabilizing controllers and observers for a class of uncertain linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, no. 9, pp. 904-907, Sep. 1985. *