異方性 Plasma 内에서 運動中인 Source에 依한 電磁界

(Electromagnetic Fields Due to Moving Sources in Anisotropic Plasma)

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要 約

異方性 plasma內를 等速運動中인 Source로 因한 電磁界理論을 一般的인 경우 즉 定磁界와 速度가 任意의 方向에 있을 경우에 대하여 論하였다. Minkowski의 關係式을 異方性分散媒体에 대해서 一般化했으며 媒体 parameter들의 相對論的變換公式을 誘導하여 速度比 β의 多項式으로 展開함으로써 利用度를 높였다. 電磁界가 滿足하는 Helmholtz의 波動方程式을 tensor parameter로 特徵지워지는 媒体에 대해 一般化했으며 이를 行列形式으로 整理하였다. 波動方程式의 解는 波動演算子行列의 逆行列과 source函數 vector의 相乘積으로 表示되며 결국 電磁界는 波動演算子行列의 逆行列에 依해서 表示된다. 波動演算子行列의 逆行列을 求하고 그 結果를 具体的으로 表示하였다. 本 論文에서 誘導한 公式들은 一般的式들로서 特殊한 條件下에서 이들은 旣知의 結果와 一致함을 例를 통해서 表示하였다.

Abstract

Fundamentals of electrodynamics of moving sources with constant velocity in an anisotropic plasma when the do magnetic field and the relative motion are oriented in arbitrary directions are presented. The well-known Minkowski's relations are generalized to accomodate anisotropic and dispersive media, and relativistic transformation formulae of constitutive parameters are derived and expanded into polynomials of the speed ratio β to increase the utility of the formulae. The helmholtz wave equation of electromagnetic fields is generalized to the media characterized by tensor parameters, and is solved in operator form. Also the solution of wave equation is expressed as a product of the inverse of the wave operator matrix and the source function vector, and the inverse of the wave operator matrix is presented in an explicit form. The equations and formulae derived in this paper are all general, and can be reduced to known and proven results upon imposing the restriction called for by specific situations.

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I. 序　論

相對論的電磁場의 理論과 應用에 관한 關心은 主로 物理學의 本質的으로는 微視的分野에 限定되는 느낌이있었다. 近者에 이르러 人工衛星과 같은 더욱 빠르고 큰 人工飛行体의 出現과 相對運動中에 있는 無線局間의 信賴할 만한 通信路建設에 必須的인 高速 plasma wind下의 電磁波輻射特性에 대한 보다 많은 智識의 必要性때문에 本質的으로 巨視的인 이 問題가 새로운 關心을 惹起시키고 있다.[1~12]

非等方性 plasma內에서의 相對運動中의 source에 의한 電磁場의 解析은 二重으로 複雜한바 그 理由는 場과 媒体가 tensor parameter를 거쳐서 相互作用하기 때문이며 또 한 理由는 電氣 vector와 磁氣 vector간에는 從來의 Faraday induction外에 새로운 型의 相互結合이 發生하기 때문이다. 이 複雜性때문에 大部分의 研究報告는 相互速度의 크기나 定磁界의 方向에 制約을 加한 比較的 簡單한 경우에 限定된 것들이다.

本 論文에서는 magneto-plasma內를 定速度로 運動中인 source로부터의 電磁界 問題를 可能한限 制約없이 가장 一般的인 형체로 解析하고자 努力하였다. 먼저 電磁場論과 Lorentz變換을 이용하여 媒体 parameter간의 Minkowsky의 關係式을 異方性媒体의 경우까지 包含되도록 一般化했으며 이를 土台로 媒体 parameter의 變換公式을 對稱型으로 誘導하였다. 다음에 電磁界가 滿足해야할 Helmholtz 波動方程式을 一般化하고 그 結果를 行列形式으로 整理하였다. 波動方程式의 解는 波動演算子行列과 source函數vector간의 相乘積으로 表示된다. Source 分布로부터 source函數vector는 곧 알 수 있으므로 電磁場을 求하는 問題는 결국 波動演算子行列의 逆行列을 求하는 問題로 歸着된다. V章에서는 이 逆行列을 구한 후 이를 이용하기 편리한 形으로 整理하였다.

II. 異方性媒体내의 電磁場論과 Lorentz 變換

電氣的 및 磁氣的 source $\underline{\mathscr{J}}(\underline{r}, t)$ 및 $\underline{\mathscr{M}}(\underline{r}, t)$가 異方性媒体내에서 等速度 $\underline{v}$로 運動中일때의 電磁界를 생각해 보자. 媒体의 電磁氣的 特性은 tensor parameter $\underline{\underline{\varepsilon}}$ 및 $\underline{\underline{\mu}}$로 표시된다. Source에 대한 靜止系를 S, 媒体에 대한 靜止系를 S′라 부르기로 하며 또 어떤 量을 표시하는데 있어서 S系에 대한 量은 그대로, S′系에 대한 것은 그 量에 ′記号를 添加 表示키로 한다.

S′系에 있어서 電磁界 $\underline{\mathscr{E}}(\underline{r}, t)$, $\underline{\mathscr{D}}(\underline{r}, t)$, $\underline{\mathscr{H}}(\underline{r}, t)$ 및 $\underline{\mathscr{B}}(\underline{r}, t)$가 만족하는 Maxwell方程式은

$$\nabla' \times \underline{\mathscr{E}}' = -\frac{\partial \underline{\mathscr{B}}'}{\partial t'} - \underline{\mathscr{M}}'; \quad \nabla' \times \underline{\mathscr{H}}' = \frac{\partial \underline{\mathscr{H}}'}{\partial t'} + \underline{\mathscr{J}}' \tag{1}$$

지금 Fourier 變換双의 定義 :

$$\underline{A}(\underline{k}, \omega) = \int_{-\infty}^{\infty} \underline{\mathscr{A}}(\underline{r}, t) e^{-j\omega t \;\; \underline{k}\cdot\underline{r}} \, d\underline{r}\, dt \tag{2}$$

$$\underline{\mathscr{A}}(\underline{r}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \underline{A}(\underline{k}, w) e^{j\omega t - \underline{k}\cdot\underline{r}} \, d\underline{k}\, d\omega \tag{3}$$

에 따라 式(1)에 Fourier 變換을 가하면

$$-j\underline{k}' \times \underline{E}' = -j\omega' \underline{B}' - \underline{M}'; \quad -j\underline{k}' \times \underline{H}' = j\omega' \underline{D}' + \underline{J}' \tag{4}$$

여기서 $\underline{E}(\underline{k}, \omega)$, $\underline{D}(\underline{k}, \omega)$, $\underline{H}(\underline{k}, \omega)$ 및 $\underline{B}(\underline{k}, \omega)$는 각각 $\underline{\mathscr{E}}(\underline{r}, t)$, $\underline{\mathscr{D}}(\underline{r}, t)$, $\underline{\mathscr{H}}(\underline{r}, t)$ 및 $\underline{\mathscr{B}}(\underline{r}, t)$, 의 Fourier 變換을 표시한다. 이들 媒体 parameter간의 關係式은

$$\underline{D}' = \varepsilon_0 \underline{\underline{\varepsilon}}' \underline{E}' \tag{5}$$

$$\underline{B}' = \mu_0 \underline{\underline{\mu}}' \underline{H}' \tag{6}$$

여기서 ε_0 및 μ_0는 自由空間 parameter들을 의미한다.

Maxwell方程式은 Lorentz 變換에 대해서 covariant property를 유지하므로 S系에 대한 Maxwell方程式은 S′系에 대한 것과 같은 形式 즉

$$\nabla \times \underline{\mathscr{E}} = -\frac{\partial \underline{\mathscr{B}}}{\partial t} - \underline{\mathscr{M}}, \quad \nabla \times \underline{\mathscr{H}} = \frac{\partial \underline{\mathscr{E}}}{\partial t} + \underline{\mathscr{J}} \tag{7}$$

이들의 Fourier 變換式은

$$-j\underline{k} \times \underline{E} = -j\omega \underline{B} - \underline{M}, \quad -j\underline{k} \times \underline{H} = j\omega \underline{D} + \underline{J} \tag{8}$$

그러나 S系로 變換했을 때의 式 (5), (6)은 전혀 다른 形式을 취하게 된다.

S系와 S′系는 Lorentz變換에 의해서 密接히 結合되어 있으며 各量에 대한 兩系간의 變換公式만 確立된다면

어떤 現象을 한系에 대해서 풀었을때 다른 系에서 그 現象이 어떻게 나타나는지 豫告할 수 있다.

任意의 電磁界 vector 예컨데 $\underline{\mathscr{E}}(\underline{r}, t)$와 그의 複素振幅vector 예컨데 $\underline{E}(\underline{k}, \omega)$에는 같은 形式의 Lorentz 變換이 적용된다.

電磁界 vector들의 Lorentz變換公式은 잘 알려진 바와 같이 [14]

$$\underline{E}' = \gamma\underline{E} + (1-\gamma)\frac{(\underline{E}\cdot\underline{V})\underline{V}}{V^2} + \gamma\underline{V}\times\underline{B} \tag{9}$$

$$\underline{B}' = \gamma\underline{B} + (1-\gamma)\frac{(\underline{B}\cdot\underline{V})\underline{V}}{V^2} - \gamma\frac{\underline{V}\times\underline{E}}{C^2} \tag{10}$$

$$\underline{D}' = \gamma\underline{D} + (1-\gamma)\frac{(\underline{D}\cdot\underline{V})\underline{V}}{V^2} + \gamma\frac{\underline{V}\times\underline{H}}{C^2} \tag{11}$$

$$\underline{H}' = \gamma\underline{H} + (1-\gamma)\frac{(\underline{H}\cdot\underline{V})\underline{V}}{V^2} - \gamma\underline{V}\times\underline{D} \tag{13}$$

여기서

$$\gamma = (1-\beta^2)^{-\frac{1}{2}} \tag{13}$$

$$\beta = \frac{V}{C} \tag{14}$$

$$C = 1/\sqrt{\varepsilon_0\mu} = \text{自由空間에서의 光速度} \tag{15}$$

한편 媒体 parameter에 대해서는 等方性媒体의 경우는 tensor parameter $\underline{\underline{\varepsilon}}'$ 및 $\underline{\underline{\mu}}'$는 scalar量 ε' 및 μ'로 縮退하며 S系에서의 該當量 ε, μ는 Minkowski의 關係式[15, 16, 17]

$$\underline{D} + \frac{1}{C^2}\underline{V}\times\underline{H} = \varepsilon_0\varepsilon'(\underline{E} + \underline{V}\times\underline{B}) \tag{16}$$

$$\underline{B} - \frac{1}{C^2}\underline{V}\times\underline{E} = \mu_0\mu'(\underline{H} - \underline{V}\times\underline{D}) \tag{17}$$

로부터 유도해낼 수 있다. 그러나 이 Minkowski의 關係式은 媒体parameter가 tensor가 되는 異方性媒体에 대해서는 그대로 적용할 수는 없다.

일반적으로 行列表記法에 의하면 2 vector간의 scalar 및 vector 積은 각각

$$\underline{A}\cdot\underline{B} = (A_1\ A_2\ A_3)\begin{vmatrix} B_1 \\ B_2 \\ B_3 \end{vmatrix} = A^T\underline{B} \tag{18}$$

$$\underline{A}\times\underline{B} = \begin{vmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{vmatrix}\begin{vmatrix} B_1 \\ B_2 \\ B_3 \end{vmatrix} = \underline{\underline{A}}\,\underline{B} \tag{19}$$

여기서

$$\underline{A}^T = \text{行列 } \underline{A}\text{의 轉置行列} \tag{20}$$

$$\underline{\underline{A}} = \begin{vmatrix} 0 & -A_3 & A_1 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{vmatrix} = \text{Vector積 } \underline{A}\times\underline{B}\text{에서의 premultiplier } \underline{\underline{A}}\text{의 等價行列} \tag{21}$$

이므로 式(9)~(12)를 行列表記할 수 있다. 예컨데 式(19)는

$$\begin{aligned}\underline{E}' &= \gamma\underline{E} + (1-\gamma)\frac{\underline{V}(\underline{V}^T\underline{E})}{V^2} + \gamma\underline{\underline{V}}\,\underline{B} \\ &= \gamma\left\{\underline{\underline{I}} + \frac{1-\gamma}{\gamma}\frac{\underline{\beta}\,\underline{\beta}^T}{\beta^2}\right\}\underline{E} + C\gamma\underline{\underline{\beta}}\,\underline{B} \\ &= \gamma\underline{\underline{U}}\,\underline{E} + C\gamma\underline{\underline{\beta}}\,\underline{B}\end{aligned} \tag{22}$$

여기서

$$\underline{\beta} = \underline{V}/C,\quad \underline{\underline{\beta}} = \underline{\underline{V}}/C \tag{23}$$

$$\underline{\underline{U}} = \underline{\underline{I}} + \frac{1-\gamma}{\gamma}\frac{\underline{\beta}\,\underline{\beta}^T}{\beta^2} \tag{24}$$

같은 방법으로 式(10), (11) 및(12)는 각각

$$\underline{B}' = \gamma \underline{\underline{U}}\,\underline{B} - \frac{1}{C}\gamma \underline{\underline{\beta}}\,\underline{E} \tag{25}$$

$$\underline{D}' = \gamma \underline{\underline{U}}\,\underline{D} + \frac{1}{C}\gamma \underline{\underline{\beta}}\,\underline{H} \tag{26}$$

$$\underline{H}' = \gamma \underline{\underline{U}}\,\underline{H} - C\gamma \underline{\underline{\beta}}\,\underline{D} \tag{27}$$

이들을 式(5), (6)에 代入하면

$$\underline{\underline{U}}\,\underline{D} + \frac{1}{C}\underline{\underline{\beta}}\,\underline{H} = \varepsilon_0 \underline{\underline{\varepsilon}}'(\underline{\underline{U}}\,\underline{E} + C\underline{\underline{\beta}}\,\underline{B}) \tag{28}$$

$$\underline{\underline{U}}\,\underline{B} - \frac{1}{C}\underline{\underline{\beta}}\,\underline{E} = \mu_0 \underline{\underline{\mu}}'(\underline{\underline{U}}\,\underline{H} - C\underline{\underline{\beta}}\,\underline{D}) \tag{29}$$

式(28), (29)가 異方性媒体에 대해서도 成立되는 一般化된 Minkowski의 關係式이다.

Ⅲ. 媒体 Parameter의 變換公式

S'系에서는 媒体 parameter $\underline{\underline{\varepsilon}}'$ 및 $\underline{\underline{\mu}}'$는 각각 電氣 vector $\underline{D}'$와 다른 電氣 vector $\underline{E}'$와를, 또는 磁氣 vector $\underline{B}'$와 다른 磁氣 vector $\underline{H}'$와를 關係지울뿐 parameter를 통한 電氣 및 磁氣 vector간의 相互結合은 存在하지 않는다. 그러나 運動中에 있는 source에 대한 靜止系 S에서는 사정이 다르게 나타난다.

1. 變換公式

式(28), (29)에서 $\underline{B}$ 또는 $\underline{D}$를 消去하면

$$\underline{D} = (\underline{\underline{U}} + \underline{\underline{\varepsilon}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\mu}}'\underline{\underline{\beta}})^{-1}\{\varepsilon_0\underline{\underline{\varepsilon}}'(\underline{\underline{U}} + \underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\beta}})\underline{E} + \frac{1}{C}(\underline{\underline{\varepsilon}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\mu}}'\underline{\underline{U}} - \underline{\underline{\beta}})\underline{H}\} \tag{30}$$

$$\underline{B} = (\underline{\underline{U}} + \underline{\underline{\mu}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\varepsilon}}'\underline{\underline{\beta}})^{-1}\{\frac{1}{C}(\underline{\underline{\beta}} - \underline{\underline{\mu}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\varepsilon}}'\underline{\underline{U}})\underline{E} + \mu_0\underline{\underline{\mu}}'(\underline{\underline{U}} + \underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\beta}})\underline{H}\} \tag{31}$$

지금

$$\underline{D} = \varepsilon_0\underline{\underline{\varepsilon}}\,\underline{E} + \underline{\underline{\xi}}\,\underline{H} \tag{32}$$

$$\underline{B} = \underline{\underline{\eta}}\,\underline{E} + \mu_0\underline{\underline{\mu}}\,\underline{H} \tag{33}$$

라 놓으면 式(32), (33)과 式(30), (31)을 비교하므로써

$$\underline{\underline{\varepsilon}} = (\underline{\underline{U}} + \underline{\underline{\varepsilon}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\mu}}'\underline{\underline{\beta}})^{-1}\underline{\underline{\varepsilon}}'(\underline{\underline{U}} + \underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\beta}}) \tag{34}$$

$$\underline{\underline{\xi}} = \frac{1}{C}(\underline{\underline{U}} + \underline{\underline{\varepsilon}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\mu}}'\underline{\underline{\beta}})^{-1}(\underline{\underline{\varepsilon}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\mu}}'\underline{\underline{U}}' - \underline{\underline{\beta}}) \tag{35}$$

$$\underline{\underline{\eta}} = \frac{1}{C}(\underline{\underline{U}} + \underline{\underline{\mu}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\varepsilon}}'\underline{\underline{\beta}})^{-1}(\underline{\underline{\beta}} - \underline{\underline{\mu}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\varepsilon}}'\underline{\underline{U}}) \tag{36}$$

$$\underline{\underline{\mu}} = (\underline{\underline{U}} + \underline{\underline{\mu}}'\underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\varepsilon}}'\underline{\underline{\beta}})^{-1}\underline{\underline{\mu}}(\underline{\underline{U}} + \underline{\underline{\beta}}\,\underline{\underline{U}}^{-1}\underline{\underline{\beta}}) \tag{37}$$

式(34)~(37)은 媒体 parameter에 대한 靜止系와 運動系間의 一般變換公式이며 Lee와 Lo[18,19]의 非對稱公式과는 달리 對稱形인 점에 주의하라.

式 (24)의 行列 $\underline{\underline{U}}$를 다시 쓰면

$$\underline{\underline{U}} = \underline{\underline{I}} - (1-\sqrt{1-\beta^2})\frac{\underline{\beta}\,\underline{\beta}^T}{\beta^2} = \underline{\underline{I}} - B_2\underline{m}\,\underline{m}^T \tag{38}$$

여기서

$$\underline{\beta} = \beta\,\underline{m} \tag{39}$$

$$\underline{m} = (m_1\ m_2\ m_3)^T = \underline{\beta}\text{의 方向餘弦} \tag{40}$$

$$B_2 = 1-\sqrt{1-\beta^2} = \frac{1}{2}\beta^2(1+\frac{1}{4}\beta^2+\frac{1}{8}\beta^2+\cdots) \tag{41}$$

따라서

$$\underline{\underline{U}}^{-1} = [\underline{\underline{I}} - B_2\underline{m}\,\underline{m}^T]^{-1}$$

$$= \frac{1}{1-B_2}\begin{bmatrix} 1-B_2(m_2^2+m_3^2) & B_2m_1m_2 & B_2m_1m_3 \\ B_2m_2m_1 & 1-B_2(m_3^2+m_1^2) & B_2m_2m_3 \\ B_2m_3m_1 & B_2m_3m_2 & 1-B_2(m_1^2+m_2^2) \end{bmatrix}$$

$$= \frac{1}{1-B_2}\{(1-B_2)\underline{\underline{I}} + B_2 \underline{m}\,\underline{m}^T\}$$

$$= \underline{\underline{I}} + \frac{B_2}{1-B_2}\underline{m}\,\underline{m}^T \tag{42}$$

式(41)의 B_2는 β^2정도의 크기이므로 대부분의 경우 式(38) 및 (42)의 第2項은 第1項에 비해서 무시해도 무방하다. 따라서 式 (34)~(37)은

$$\underline{\underline{\varepsilon}} = (\underline{\underline{I}} + \underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}}\,\underline{\underline{\mu}}'\,\underline{\underline{\beta}})^{-1}\underline{\underline{\varepsilon}}'(\underline{\underline{I}} + \underline{\underline{\beta}}\,\underline{\underline{\beta}}) \tag{43}$$

$$\underline{\underline{\zeta}} = \frac{1}{C}(\underline{\underline{I}} + \underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}}\,\underline{\underline{\mu}}'\,\underline{\underline{\beta}})^{-1}(\underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}}\,\underline{\underline{\mu}}' - \underline{\underline{\beta}}) \tag{44}$$

$$\eta = \frac{1}{C}(\underline{\underline{I}} + \underline{\underline{\mu}}'\,\underline{\underline{\beta}}\,\underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}})^{-1}(\underline{\underline{\beta}} - \underline{\underline{\mu}}'\,\underline{\underline{\beta}}\,\underline{\underline{\varepsilon}}') \tag{45}$$

$$\underline{\underline{\mu}} = (\underline{\underline{I}} + \underline{\underline{\mu}}'\,\underline{\underline{\beta}}\,\underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}})^{-1}\underline{\underline{\mu}}'(\underline{\underline{I}} + \underline{\underline{\beta}}\,\underline{\underline{\beta}}) \tag{46}$$

Magneto-plasma 또는 gyromagnetic plasma에 있어서는 $\underline{\underline{\mu}}' = \underline{\underline{I}}$ 이므로 式(43)~(46)은 더욱 간단해져서

$$\underline{\underline{\varepsilon}} = (\underline{\underline{I}} + \underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}}\,\underline{\underline{\beta}})^{-1}\underline{\underline{\varepsilon}}'(\underline{\underline{I}} + \underline{\underline{\beta}}\,\underline{\underline{\beta}}) \tag{47}$$

$$\underline{\underline{\zeta}} = \frac{1}{C}(\underline{\underline{I}} + \underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}}\,\underline{\underline{\beta}})^{-1}(\underline{\underline{\varepsilon}}' - \underline{\underline{I}})\underline{\underline{\beta}} \tag{48}$$

$$\underline{\underline{\eta}} = \frac{1}{C}(\underline{\underline{I}} + \underline{\underline{\beta}}\,\underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}})^{-1}\underline{\underline{\beta}}(\underline{\underline{I}} - \underline{\underline{\varepsilon}}') \tag{49}$$

$$\underline{\underline{\mu}} = (\underline{\underline{I}} + \underline{\underline{\beta}}\,\underline{\underline{\varepsilon}}'\beta)^{-1}(\underline{\underline{I}} + \underline{\underline{\beta}}\,\underline{\underline{\beta}}) \tag{50}$$

2. 變換公式의 多項式展開

β의 값은 1보다 작은 것이 보통이므로 式(47)~(50)을 β의 多項式으로 展開할 수 있다면 편리할 것이다. 式(47)~(50)을 展開하기 전에 式들에 나타난 逆行列들의 要素를 계산하였으며 그 결과는 다음과 같다.

지금

$$({}_1\triangle)^{-1} \equiv (\underline{\underline{I}} + \underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}}\,\underline{\underline{\beta}})^{-1} = \frac{\mathrm{adj}({}_1\triangle)}{\det({}_1\triangle)} = \frac{[\mathrm{cof}({}_1\triangle_{ji})]}{\det({}_1\triangle)} \tag{51}$$

라 놓면 다음과 같이 된다.

$$\mathrm{cof}({}_1\triangle_{ii}) = \beta^4 m_i \underline{m}^T(E_{i1}, E_{i2}, E_{i3})^T + \beta^2\{m_j^2\varepsilon'_{jj} + m_k^2\varepsilon'_{kk} + m_i m_j \varepsilon'_{ji} + m_i m_k \varepsilon'_{ki} + m_j m_k(\varepsilon'_{jk} + \varepsilon'_{kj}) - (\varepsilon'_{jj} + \varepsilon'_{kk})\} + 1, \tag{52}$$

$$\mathrm{cof}({}_1\triangle_{ij})\ (i \neq j) = \beta^4 m_j \underline{m}^T(E_{i1}, E_{i2}, E_{i3})^T - \beta^2\{m_i^2\varepsilon'_{ji} + m_i m_j \varepsilon'_{jj} + m_i m_k \varepsilon'_{jk}) - \varepsilon'_{ji}\}, \tag{53}$$

$$\det({}_1\triangle) = \underline{m}^T\left\{\beta^4\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33}\end{bmatrix} + \beta^2\begin{bmatrix} -(\varepsilon'_{22}+\varepsilon'_{33}) & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{21} & -(\varepsilon'_{33}+\varepsilon'_{11}) & \varepsilon'_{23} \\ \varepsilon'_{31} & \varepsilon'_{32} & -(\varepsilon'_{11}+\varepsilon'_{22})\end{bmatrix} + \underline{\underline{I}}\right\}\underline{m} \tag{54}$$

여기서 $E_{ij}=\det(\underline{\underline{\varepsilon}})$의 要素 ε_{ij}에 관한 餘因子 (55)

式 (52), (53)으로부터

$$\mathrm{adj}({}_1\triangle) = \beta^4\underline{m}\,\underline{m}^T\begin{bmatrix} E_{11} & E_{21} & E_{31} \\ E_{12} & E_{22} & E_{32} \\ E_{13} & E_{23} & E_{33}\end{bmatrix} + \beta^2\left[\begin{bmatrix} -(\varepsilon'_{22}+\varepsilon'_{33}) & \varepsilon'_{21} & \varepsilon'_{31} \\ \varepsilon'_{12} & -(\varepsilon'_{33}+\varepsilon'_{11}) & \varepsilon'_{32} \\ \varepsilon'_{13} & \varepsilon'_{23} & -(\varepsilon'_{11}+\varepsilon'_{22})\end{bmatrix}\right.$$

$$\left. - \begin{bmatrix} -\underline{m}^T\begin{bmatrix} 0 & 0 & 0 \\ \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23} \\ \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33}\end{bmatrix}\underline{m} & \underline{m}^T\begin{bmatrix} 0 & 0 & 0 \\ \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ 0 & 0 & 0\end{bmatrix}\underline{m} & \underline{m}^T\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13}\end{bmatrix}\underline{m} \\ \underline{m}^T\begin{bmatrix} \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\underline{m} & -\underline{m}^T\begin{bmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ 0 & 0 & 0 \\ \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33}\end{bmatrix}\underline{m} & \underline{m}^T\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23}\end{bmatrix}\underline{m} \\ \underline{m}^T\begin{bmatrix} \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\underline{m} & \underline{m}^T\begin{bmatrix} 0 & 0 & 0 \\ \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33} \\ 0 & 0 & 0\end{bmatrix}\underline{m} & -\underline{m}^T\begin{bmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23} \\ 0 & 0 & 0\end{bmatrix}\underline{m}\end{bmatrix}\right] \tag{56}$$

또

$$({}_2\triangle)^{-1} \equiv (\underline{\underline{I}} + \underline{\underline{\beta}}\,\underline{\underline{\varepsilon}}'\,\underline{\underline{\beta}})^{-1} = \frac{\mathrm{adj}({}_2\triangle)}{\det({}_2\triangle)} = \frac{[\mathrm{cof}({}_2\triangle_{ji})]}{\det({}_2\triangle)} \tag{57}$$

라 놓면 여기서

$$\mathrm{cof}({}_2\triangle_{ii}) = \beta^4 m_i^2 \underline{m}^T \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \underline{m} - \beta^2 \{ m_i^2 (\varepsilon'_{jj} + \varepsilon'_{kk}) + (m_j^2 + m_k^2)\varepsilon'_{ii} - m_i m_j (\varepsilon'_{ij} + \varepsilon'_{ji})$$

$$- m_i m_k (\varepsilon'_{ik} + \varepsilon'_{ki}) \} + 1 \tag{58}$$

$$\mathrm{cof}({}_2\triangle_{ij})\ (i \neq j) = \beta^4 m_i m_j \underline{m}^T \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \underline{m} - \beta^2 \{ m_k^2 \varepsilon'_{ij} + m_i m_j \varepsilon'_{kk} - m_i m_k \varepsilon'_{kj} - m_j m_k \varepsilon'_{ik} \} \tag{59}$$

$$\det({}_2\triangle) = \underline{m}^T \left\{ \beta^4 \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{13} \\ E_{31} & E_{32} & E_{13} \end{bmatrix} + \beta^2 \begin{bmatrix} -(\varepsilon'_{22} + \varepsilon'_{33}) & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{21} & -(\varepsilon'_{33} + \varepsilon'_{11}) & \varepsilon'_{23} \\ \varepsilon'_{31} & \varepsilon'_{32} & -(\varepsilon'_{11} + \varepsilon'_{22}) \end{bmatrix} + \underline{\underline{I}} \right\} \underline{m} \tag{60}$$

式 (58), (59) 로부터

$$\mathrm{adj}({}_2\triangle) = \beta^4 \underline{m}\,\underline{m}^T \left\{ \underline{m}^T \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \underline{m} \right\}$$

$$+\beta^2 \begin{bmatrix} \underline{m}^T \begin{bmatrix} -(\varepsilon'_{22}+\varepsilon'_{33}) & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{21} & -\varepsilon'_{11} & 0 \\ \varepsilon'_{31} & 0 & -\varepsilon'_{11} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} 0 & 0 & 0 \\ -\varepsilon'_{33} & 0 & \varepsilon'_{31} \\ \varepsilon'_{23} & 0 & -\varepsilon'_{21} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} 0 & 0 & 0 \\ \varepsilon'_{32} & -\varepsilon'_{31} & 0 \\ -\varepsilon'_{22} & \varepsilon'_{21} & 0 \end{bmatrix} \underline{m} \\ \underline{m}^T \begin{bmatrix} 0 & -\varepsilon'_{33} & \varepsilon'_{32} \\ 0 & 0 & 0 \\ 0 & \varepsilon'_{13} & -\varepsilon'_{12} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} -\varepsilon'_{22} & \varepsilon'_{12} & 0 \\ \varepsilon'_{21} & -(\varepsilon'_{33}+\varepsilon'_{11}) & \varepsilon'_{23} \\ 0 & \varepsilon'_{32} & -\varepsilon'_{22} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} -\varepsilon'_{32} & \varepsilon'_{31} & 0 \\ 0 & \varepsilon'_{31} & 0 \\ \varepsilon'_{12} & -\varepsilon'_{11} & 0 \end{bmatrix} \underline{m} \\ \underline{m}^T \begin{bmatrix} 0 & \varepsilon'_{23} & -\varepsilon'_{22} \\ 0 & -\varepsilon'_{13} & \varepsilon'_{12} \\ 0 & 0 & 0 \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} -\varepsilon'_{23} & 0 & \varepsilon'_{21} \\ \varepsilon'_{13} & 0 & -\varepsilon'_{11} \\ 0 & 0 & 0 \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} -\varepsilon'_{33} & 0 & \varepsilon'_{13} \\ 0 & -\varepsilon'_{33} & \varepsilon'_{23} \\ \varepsilon'_{31} & \varepsilon'_{32} & -(\varepsilon'_{11}+\varepsilon'_{22}) \end{bmatrix} m \end{bmatrix}$$

$$+\underline{\underline{I}} \tag{61}$$

式 (51)～(61)을 式 (47)～(50)에 代入하므로써 S系에서의 媒体 parameter들을 다음과 같이 β의 多項式으로 展開할 수 있다.

$$= \frac{1}{\det({}_1\triangle)} \left[\underline{\underline{\varepsilon}}' - \beta^2 \left\{ \underline{\underline{\varepsilon}}' (\underline{\underline{I}} - \underline{m}\,\underline{m}^T) + \begin{bmatrix} E_{22}+E_{33} & -E_{21} & -E_{31} \\ -E_{12} & E_{33}+E_{11} & -E_{32} \\ -E_{13} & -E_{23} & E_{11}+E_{22} \end{bmatrix} \right. \right.$$

$$+ \begin{bmatrix} -\underline{m}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_{33} & E_{32} \\ 0 & -E_{23} & E_{22} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} 0 & 0 & 0 \\ E_{33} & 0 & -E_{31} \\ -E_{23} & 0 & E_{21} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} 0 & 0 & 0 \\ -E_{32} & E_{31} & 0 \\ E_{32} & -E_{21} & 0 \end{bmatrix} \underline{m} \\ \underline{m}^T \begin{bmatrix} 0 & E_{33} & -E_{32} \\ 0 & 0 & 0 \\ 0 & -E_{13} & E_{12} \end{bmatrix} \underline{m} & -\underline{m}^T \begin{bmatrix} E_{33} & 0 & -E_{31} \\ 0 & 0 & 0 \\ -E_{13} & 0 & E_{11} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} E_{32} & -E_{31} & 0 \\ 0 & 0 & 0 \\ -E_{12} & E_{11} & 0 \end{bmatrix} \underline{m} \\ \underline{m}^T \begin{bmatrix} 0 & -E_{23} & E_{22} \\ 0 & E_{13} & -E_{12} \\ 0 & 0 & 0 \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} E_{23} & 0 & -E_{21} \\ -E_{13} & 0 & E_{11} \\ 0 & 0 & 0 \end{bmatrix} \underline{m} & -\underline{m}^T \begin{bmatrix} E_{22} & -E_{21} & 0 \\ -E_{12} & E_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{m} \end{bmatrix}$$

$$+\beta^4 \left\{ \det(\underline{\underline{\varepsilon}}')\, \underline{m}\,\underline{m}^T + \begin{bmatrix} E_{22}+E_{33} & -E_{21} & -E_{31} \\ -E_{12} & E_{33}+E_{11} & -E_{32} \\ -E_{13} & -E_{23} & E_{11}+E_{22} \end{bmatrix} (\underline{\underline{I}} - \underline{m}\,\underline{m}^T) \right.$$

$$+ \begin{bmatrix} -\underline{m}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_{23} & -E_{32} \\ 0 & -E_{23} & E_{22} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} 0 & 0 & 0 \\ E_{33} & 0 & -E_{31} \\ -E_{23} & 0 & E_{21} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} 0 & 0 & 0 \\ -E_{32} & E_{31} & 0 \\ E_{22} & -E_{21} & 0 \end{bmatrix} \underline{m} \\ \underline{m}^T \begin{bmatrix} 0 & E_{33} & -E_{32} \\ 0 & 0 & 0 \\ 0 & -E_{13} & E_{12} \end{bmatrix} \underline{m} & -\underline{m}^T \begin{bmatrix} E_{33} & 0 & -E_{31} \\ 0 & 0 & 0 \\ -E_{13} & 0 & E_{11} \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} E_{32} & -E_{31} & 0 \\ 0 & 0 & 0 \\ -E_{12} & E_{11} & 0 \end{bmatrix} \underline{m} \\ \underline{m}^T \begin{bmatrix} 0 & -E_{23} & E_{22} \\ 0 & E_{13} & -E_{12} \\ 0 & 0 & 0 \end{bmatrix} \underline{m} & \underline{m}^T \begin{bmatrix} E_{23} & 0 & -E_{21} \\ -E_{12} & 0 & E_{11} \\ 0 & 0 & 0 \end{bmatrix} \underline{m} & -\underline{m}^T \begin{bmatrix} E_{22} & -E_{31} & 0 \\ -E_{12} & E_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{m} \end{bmatrix}$$

$$-\beta^6 \det(\underline{\underline{\varepsilon}}')\ (\underline{\underline{I}} - \underline{m}\,\underline{m}^T) \} \tag{62}$$

$$\underline{\mu}=\frac{1}{\det({}_2\triangle)}\Bigg[\underline{I}-\beta^2\{(\underline{I}-\underline{m}\,\underline{m}^T)$$

$$+\left[\begin{array}{lll}
\underline{m}^T\begin{bmatrix}\varepsilon_{22}+\varepsilon_{33} & -\varepsilon_{12} & -\varepsilon_{13}\\ -\varepsilon_{21} & \varepsilon_{11} & 0\\ -\varepsilon_{31} & 0 & \varepsilon_{11}\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}0 & 0 & 0\\ \varepsilon_{33} & 0 & -\varepsilon_{31}\\ -\varepsilon_{23} & 0 & \varepsilon_{21}\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}0 & 0 & 0\\ -\varepsilon_{32} & \varepsilon_{31} & 0\\ e_2 & -\varepsilon_{21} & 0\end{bmatrix}\underline{m}\\
\underline{m}^T\begin{bmatrix}0 & \varepsilon_{33} & -\varepsilon_{32}\\ & 0 & 0\\ & -\varepsilon_{13} & \varepsilon_{12}\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}\varepsilon_{22} & -\varepsilon_{12} & 0\\ -\varepsilon_{21} & \varepsilon_{33}+\varepsilon_{11} & -\varepsilon_{23}\\ 0 & -\varepsilon_{32} & \varepsilon_{22}\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}\varepsilon_{32} & -\varepsilon_{31} & 0\\ 0 & 0 & 0\\ -\varepsilon_{12} & \varepsilon_{11} & 0\end{bmatrix}\underline{m}\\
\underline{m}^T\begin{bmatrix}0 & -\varepsilon_{23} & \varepsilon_{22}\\ 0 & \varepsilon_{13} & -\varepsilon_{12}\\ 0 & 0 & 0\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}\varepsilon_{23} & 0 & -\varepsilon_{21}\\ -\varepsilon_{21} & 0 & \varepsilon_{11}\\ 0 & 0 & 0\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}0 & 0 & -\varepsilon_{13}\\ -0 & \varepsilon_{33} & -\varepsilon_{23}\\ -\varepsilon_{31} & -\varepsilon_{32} & \varepsilon_{11}+\varepsilon_{22}\end{bmatrix}\underline{m}
\end{array}\right]$$

$$+\beta_4\left\{\begin{array}{lll}
\underline{m}^T\begin{bmatrix}\varepsilon_{22}+\varepsilon_{33} & -\varepsilon_{12} & -\varepsilon_{13}\\ -\varepsilon_{21} & \varepsilon_{11} & 0\\ -\varepsilon_{31} & 0 & \varepsilon_{11}\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}0 & 0 & 0\\ \varepsilon_{33} & 0 & -\varepsilon_{31}\\ -\varepsilon_{23} & 0 & \varepsilon_{21}\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}0 & 0 & 0\\ -\varepsilon_{32} & \varepsilon_{33} & 0\\ \varepsilon_{22} & -\varepsilon_{32} & 0\end{bmatrix}\underline{m}\\
\underline{m}^T\begin{bmatrix}0 & \varepsilon_{33} & -\varepsilon_{32}\\ 0 & 0 & 0\\ 0 & -\varepsilon_{13} & \varepsilon_{12}\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}\varepsilon_{22} & -\varepsilon_{12} & 0\\ -\varepsilon_{21} & \varepsilon_{33}+\varepsilon_{11} & -\varepsilon_{23}\\ 0 & -\varepsilon_{32} & \varepsilon_{22}\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}\varepsilon_{32} & -\varepsilon_{31} & 0\\ 0 & 0 & 0\\ -\varepsilon_{12} & \varepsilon_{11} & 0\end{bmatrix}\underline{m}\\
\underline{m}^T\begin{bmatrix}0 & -\varepsilon_{23} & \varepsilon_{12}\\ 0 & \varepsilon_{13} & -\varepsilon_{12}\\ 0 & 0 & 0\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}\varepsilon_{23} & 0 & -\varepsilon_{21}\\ -\varepsilon_{13} & 0 & \varepsilon_{11}\\ 0 & 0 & 0\end{bmatrix}\underline{m} &
\underline{m}^T\begin{bmatrix}\varepsilon_{33} & 0 & -\varepsilon_{13}\\ 0 & \varepsilon_{33} & -\varepsilon_{23}\\ -\varepsilon_{31} & -\varepsilon_{32}\,\varepsilon_{11} & +\varepsilon_{22}\end{bmatrix}\underline{m}
\end{array}\right\}(\mathrm{I}-\underline{m}\,\underline{m}^T)$$

$$+\left\{\underline{m}^T\begin{bmatrix}E_{11} & E_{12} & E_{13}\\ E_{21} & E_{22} & E_{23}\\ E_{31} & E_{32} & E_{33}\end{bmatrix}\underline{m}\right\}\underline{m}\,\underline{m}^T\Bigg\}\Bigg] \tag{63}$$

$$\underline{\zeta}=\frac{\beta/C}{\det({}_1\triangle)}\Bigg[(\underline{\varepsilon}'-\underline{I})\begin{bmatrix}0 & -m_3 & m_2\\ m_3 & 0 & -m_1\\ -m_2 & m_1 & 0\end{bmatrix}$$

$$+\beta^2\left\{\begin{bmatrix}m_2D_{31}-m_3D_{21} & m_3D_{11}-m_1D_{31} & m_1D_{21}-m_2D_{11}\\ m_2D_{32}-m_3D_{22} & m_3D_{12}-m_1D_{32} & m_1D_{22}-m_2D_{12}\\ m_2D_{33}-m_3D_{23} & m_3D_{13}-m_1D_{33} & m_1D_{23}-m_2D_{13}\end{bmatrix}-\begin{bmatrix}m_2A_{33}-m_3A_{22} & m_3C_{31}-m_1B_{31} & m_1B_{21}-m_2C_{21}\\ m_2B_{32}-m_3C_{22} & m_3A_{31}-m_1A_{13} & m_1C_{12}-m_2B_{12}\\ m_2C_{23}-m_3B_{23} & m_3B_{13}-m_1C_{13} & m_1A_{12}-m_2A_{21}\end{bmatrix}\right\}$$

$$+\beta^4\underline{m}\left(\underline{m}^T\begin{bmatrix}0 & 0 & 0\\ E_{31} & E_{32} & E_{33}\\ -E_{21} & -E_{22} & -E_{23}\end{bmatrix}\underline{m},\ \underline{m}^T\begin{bmatrix}-E_{31} & -E_{32} & -E_{33}\\ 0 & 0 & 0\\ E_{11} & E_{12} & E_{13}\end{bmatrix}\underline{m},\ \underline{m}^T\begin{bmatrix}E_{21} & E_{22} & E_{23}\\ -E_{11} & -E_{12} & -E_{13}\\ 0 & 0 & 0\end{bmatrix}\underline{m}\right)\Bigg] \tag{64}$$

여기서

$$A_{ij}=(1-m_k^2)E_{jk}+m_im_kE_{ji}+m_jm_kE_{jj}$$

$$B_{ij}=m_i^2(\varepsilon'_{ij}+\varepsilon'_{ji})-m_k^2E_{ij}+m_im_k(E_{kj}+\varepsilon'_{jk}+\varepsilon'_{kj})+m_jm_kE_{ik}$$

$$C_{ij}=m_i^2(E_{kk}-\varepsilon'_{ii})+m_j^2(E_{kk}-\varepsilon'_{jj})+m_k^2(E_{ii}-\varepsilon'_{kk})-m_im_k(E_{ik}+E_{ki}+\varepsilon'_{ik}+\varepsilon'_{ki})$$

$$D_{ii}=(\varepsilon'_{ii}-1)(\varepsilon'_{jj}+\varepsilon'_{kk})-\varepsilon'^2_{ji}-\varepsilon'^2_{ki}$$

$$D_{ij}(i\neq j)=\varepsilon'_{ii}(\varepsilon'_{ji}-\varepsilon'_{ij})+\varepsilon'_{ji}\varepsilon'_{kk}-\varepsilon'_{ki}\varepsilon'_{kj}+\varepsilon'_{ij} \tag{65}$$

$$\underline{\eta}=\frac{\beta/C}{\det({}_2\triangle)}\left\{\begin{bmatrix}0 & -m_3 & m_2\\ m_3 & 0 & -m_1\\ -m_2 & m_1 & 0\end{bmatrix}(\underline{I}-\underline{\varepsilon}')\right.$$

$$\left.+\beta^2\begin{bmatrix}m_3F_{12}-m_2F_{13} & -(m_3I_{12}+m_2G_{12}) & m_2G_{13}+m_3I_{13}\\ m_1G_{21}+m_3I_{31} & m_1F_{23}-m_3F_{21} & -(m_1I_{23}+m_3G_{23})\\ -(m_2I_{31}+m_1G_{31}) & m_2G_{32}+m_1I_{32} & m_2F_{31}-m_1F_{32}\end{bmatrix}\right\} \tag{66}$$

여기서

$$F_{ij}=m_i^2E_{ij}-(m_j^2+m_k^2)\varepsilon'_{ji}+m_im_j(E_{jj}+\varepsilon'_{jj})+m_im_k(E_{ki}+\varepsilon'_{jk})$$

$$G_{ij}=-m_i^2\varepsilon'_{kj}+(m_j^2+m_k^2)E_{jk}+m_im_j(E_{ik}+\varepsilon'_{ki})$$

$$I_{ij}=m_j^2(E_{ii}-\varepsilon'_{kk})+(m_j^2+m_k^2)(E_{kk}-\varepsilon'_{ii})+m_im_j(E_{ji}+\varepsilon'_{ij})+m_im_k(E_{ik}+E_{ki}+\varepsilon'_{ik}+\varepsilon'_{ki}) \tag{67}$$

3. 例　題

以上에서 얻은 公式들을 特殊한 경우에 대해서 적용해 보자. 第1例로서 source와 媒体간에 相對運動이 存在하지 않은 경우는 $\beta=0$이므로 式(54), (60)로부터 $\det({}_1\triangle)=\det({}_2\triangle)=1$. 따라서 式(62)~(67)로 부터

$$\underline{\underline{\varepsilon}}=\underline{\underline{\varepsilon}}',\quad \underline{\underline{\mu}}=\underline{\underline{I}},\quad \underline{\underline{\zeta}}=0,\quad \underline{\underline{\eta}}=0 \tag{68}$$

즉 우리가 豫測했던대로이다.

第2例로 相對運動은 存在하나 運動速度가 작아서 β^2以上의 高次項을 무시할 수 있을 경우에는 式(54), (60)로부터 $\det({}_1\triangle)=\det({}_2\triangle)=1$이므로 式 (62)~(67)로부터

$$\left.\begin{aligned}
&\underline{\underline{\varepsilon}}=\underline{\underline{\varepsilon}}'\\
&\underline{\underline{\mu}}=\underline{\underline{I}}\\
&\underline{\underline{\zeta}}=\frac{\beta}{C}(\underline{\underline{\varepsilon}}'-\underline{\underline{I}})\begin{bmatrix}0 & -m_3 & m_2\\ m_3 & 0 & -m_1\\ -m_2 & m_1 & 0\end{bmatrix}=\varepsilon_0\mu_0\underline{\underline{\varepsilon}}'\underline{V}\times-\frac{1}{C^2}\underline{V}\times\\
&\underline{\underline{\eta}}=\frac{\beta}{C}\begin{bmatrix}0 & -m_3 & m_2\\ m_3 & 0 & -m_1\\ -m_2 & m_1 & 0\end{bmatrix}(\underline{\underline{I}}-\underline{\underline{\varepsilon}}')=-\varepsilon_0\mu_0\underline{V}\times\underline{\underline{\varepsilon}}'+\frac{1}{C^2}\underline{V}\times
\end{aligned}\right\} \tag{69}$$

(69)式은 Tai[20]가 얻은 一次近似理論의 結果와 一致한다.

Ⅳ. 運動中의 Source로 因한 電磁界

($\underline{k}$, ω)領域에서의 Maxwell方程式은 式(8)에 표시한 바와 같이

$$-j\underline{\underline{k}}\,\underline{E}=-j\omega\underline{B}-\underline{M} \tag{70}$$

$$-j\underline{\underline{k}}\,\underline{H}=j\omega\underline{D}+\underline{J} \tag{71}$$

式(32), (33)을 式(70),(71)에 代入整理하면

$$\underline{H}=j\frac{1}{\omega\mu_0}\underline{\underline{\mu}}^{-1}\underline{M}+\frac{1}{\omega\mu_0}\underline{\underline{\mu}}^{-1}(\underline{\underline{k}}-\omega\underline{\underline{\eta}})\underline{E} \tag{72}$$

$$\underline{E}=j\frac{1}{\omega\varepsilon_0}\underline{\underline{\varepsilon}}^{-1}\underline{J}-\frac{1}{\omega\varepsilon_0}\underline{\underline{\varepsilon}}^{-1}(\underline{\underline{k}}+\omega\underline{\underline{\zeta}})\underline{H} \tag{73}$$

式(72), (73)에서 $\underline{H}(\underline{k},\ \omega)$ 또는 $\underline{E}(\underline{k},\ \omega)$를 消去하면 ($\underline{k}$, ω)領域에서 다음과 같은 Helmholtz의 波動方程式을 얻는다.

$$\underline{\underline{W}}_E\underline{E}=\underline{S}_E \tag{74}$$

$$\underline{\underline{W}}_M\underline{H}=\underline{S}_M \tag{75}$$

여기서 波動演算子行列 (wave operator matrix) $W_E(\underline{k},\ \omega)$, $W_M(\underline{k},\ \omega)$는 각각

$$\underline{\underline{W}}_E=k_0^2\underline{\underline{\varepsilon}}+(\underline{\underline{k}}+\omega\underline{\underline{\zeta}})\underline{\underline{\mu}}^{-1}(\underline{\underline{k}}-\omega\underline{\underline{\eta}}) \tag{76}$$

$$\underline{\underline{W}}_M=k_0^2\underline{\underline{\mu}}+(\underline{\underline{k}}-\omega\underline{\underline{\eta}})\underline{\underline{\varepsilon}}^{-1}(\underline{\underline{k}}+\omega\underline{\underline{\zeta}}) \tag{77}$$

source函數 vector $\underline{S}_E(\underline{k},\ \omega)$, $\underline{S}_M(\underline{k},\ \omega)$는 각각

$$\underline{S}_E=j\omega\mu_0\underline{J}-j(\underline{\underline{k}}+\omega\underline{\underline{\zeta}})\underline{\underline{\mu}}^{-1}\underline{M} \tag{78}$$

$$\underline{S}_M=j\omega\varepsilon_0\underline{M}+j(\underline{\underline{k}}-\omega\underline{\underline{\eta}})\underline{\underline{\varepsilon}}^{-1}\underline{J} \tag{79}$$

式 (74), (75)를 $\underline{E}(\underline{k},\ \omega)$ 또는 $\underline{H}(\underline{k},\ \omega)$에 대해서 풀면

$$\underline{E}=\underline{\underline{W}}_E^{-1}\underline{S}_E \tag{80}$$

$$\underline{H}=\underline{\underline{W}}_M^{-1}\underline{S}_M \tag{81}$$

만약 Source vector $\underline{S}_E(\underline{k},\ \omega)$가 주어졌을 때는 우선 電界 $\underline{E}(\underline{k},\ \omega)$를 式 (80)로부터 求한다음 磁界$\underline{H}(\underline{k},\ \omega)$는 式(72)으로 求할 수 있다. 또 만약 source vector $\underline{S}_M(\underline{k},\ \omega)$가 주어졌다면 磁界$\underline{H}(\underline{k},\ \omega)$를 式(81)에서 求한 다음 電界$\underline{E}(\underline{k},\ \omega)$는 式 (73)으로 求할 수 있다.

Ⅴ. 波動演算子行列과 逆行列

Source 函數가 주어졌을 때에는 式(80), (81)에 의하여 電磁界는 波動演算子行列의 逆行列과 source函數 vector의 相乘積으로 표시되므로 電磁界를 求하는 문제는 波動行列의 逆行列을 求하는 문제로 귀착된다.

1. 波動演算子行列

式(76)에 式 (62)~(67)을 代入하여 整理하면

$$\underline{\underline{W}}_E(\underline{k}) = (W_{Eij}) = \frac{1}{\det(\underline{\underline{\mu}})}\left[k^2\underline{\underline{J}} + k\omega\left\{\begin{bmatrix} n_2K_{31}-n_3K_{21} & n_2K_{32}-n_3K_{22} & n_2K_{33}-n_3K_{23} \\ n_3K_{11}-n_1K_{31} & n_3K_{12}-n_1K_{32} & n_3K_{13}-n_1K_{33} \\ n_1K_{21}-n_2K_{11} & n_1K_{22}-n_2K_{12} & n_1K_{23}-n_2K_{13} \end{bmatrix}\right.\right.$$

$$\left.\left. + \begin{bmatrix} n_2L_{13}-n_3L_{12} & n_3L_{11}-n_1L_{13} & n_1L_{12}-n_2L_{11} \\ n_2L_{23}-n_3L_{22} & n_3L_{21}-n_1L_{23} & n_1L_{22}-n_2L_{21} \\ n_2L_{33}-n_3L_{32} & n_3L_{31}-n_1L_{33} & n_1L_{32}-n_2L_{31} \end{bmatrix}\right\} - \omega^2\underline{\underline{N}} \right] + k_0^2\underline{\underline{\varepsilon}} \tag{82}$$

$$J_{ii} = -n_j^2M_{kk} - n_k^2M_{jj} + n_jn_k(M_{jk}+M_{kj}) \tag{83}$$

$$J_{ij}(i\neq j) = n_k^2M_{ij} + n_in_jM_{kk} - n_in_kM_{kj} - n_jn_kM_{ik} \tag{84}$$

$$K_{ij} = (M_{1i},\ M_{2i},\ M_{3i})(\eta_{1j},\ \eta_{2j},\ \eta_{3j})^T \tag{86}$$

$$L_{ij} = (\zeta_{i1},\ \zeta_{i2},\ \zeta_{i3})(M_{j1},\ M_{j2},\ M_{j3})^T \tag{87}$$

$$N_{ij} = (\zeta_{i1},\ \zeta_{i2},\ \zeta_{i3})(M)^T(\eta_{1j},\ \eta_{2j},\ \eta_{3j})^T \tag{88}$$

$$(M) = (M_{ij}) = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \tag{89}$$

$$M_{ij} = \text{cofactor of the element } \mu_{ij} \text{ of } \det(\underline{\underline{\mu}}) \tag{90}$$

2. 電氣波動行列의 逆行列

波動行列의 逆行列를 求하는 과정은 대단히 複雜하고도 지루하므로 여기서는 그 結果만을 표시하겠다. 지금

$$\underline{\underline{W}}_E^{-1} = \frac{\mathrm{adj}(\underline{\underline{W}}_E)}{\det(\underline{\underline{W}}_E)} = \frac{[\mathrm{cof}(W_{Eji})]}{\det(\underline{\underline{W}}_E)} \tag{91}$$

라 놓면 각 餘因子는 다음과 같은 k의 四次多項式이 된다.

$$\mathrm{cof}(W_{Eij}) = k^4\left(\frac{1}{\det(\underline{\underline{\mu}})}\right)^2 n_in_j\underline{n}^T(\mathscr{M})\underline{n} + k^3\omega\left(\frac{1}{\det(\underline{\underline{\mu}})}\right)^2\{P_{ij}+Q_{ij}\} + k^2\left(\frac{1}{\det(\underline{\underline{\mu}})}\right)\left\{\frac{\omega^2}{\det(\underline{\underline{\mu}})}R_{ij} + k_0^2S_{ij}\right\}$$

$$+ k\omega\left(\frac{1}{\det(\underline{\underline{\mu}})}\right)\left\{\frac{\omega^2}{\det(\underline{\underline{\mu}})}U_{ij} + k_0^2V_{ij}\right\} + \omega^2\left(\frac{1}{\det(\underline{\underline{\mu}})}\right)\left\{\frac{\omega^2}{\det(\underline{\underline{\mu}})}(\mathscr{H}_{1j}\mathscr{H}_{2j}\mathscr{H}_{3j})(\mathscr{M})(Z_{i1}Z_{i2}Z_{i3})^T + k_0^2T_{ij}\right\}$$

$$+ k_0^4E_{ij} \tag{92}$$

여기서

$$(E) = (E_{ij}) = (\text{cofactor of the element } \varepsilon_{ij} \text{ of } \det(\underline{\underline{\varepsilon}}))$$

$$(M) = (M_{ij}) = (\text{cofactor of the element } \mu_{ij} \text{ of } \det(\underline{\underline{\mu}}))$$

$$(\varepsilon) = (\varepsilon_{ij}) = (\text{cofactor of the element } E_{ij} \text{ of } \det(\underline{\underline{E}}))$$

$$(\mathscr{M}) = (\mathscr{M}_{ij}) = (\text{cofactor of the element } M_{ij} \text{ of } \det(\underline{\underline{M}}))$$

$$(\mathscr{H}) = (\mathscr{H}_{ij}) = (\text{cofactor of the element } \eta_{ij} \text{ of } \det(\underline{\underline{\eta}}))$$

$$(Z) = (Z_{ij}) = (\text{cofactor of the element } \zeta_{ij} \text{ of } \det(\underline{\underline{\zeta}})) \tag{93}$$

$$P_{ii} = n_i(n_in_jn_k)\left\{\begin{bmatrix} 0 & 0 & 0 \\ \eta_{ij} & \eta_{jj} & \eta_{ki} \\ \eta_{ik} & \eta_{jk} & \eta_{kk} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ -\mu_{ki} & -\mu_{ki} & -\mu_{kk} \\ \mu_{ji} & \mu_{jj} & \mu_{jk} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \eta_{ik} & \eta_{ik} & \eta_{kk} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mu_{ji} & \mu_{jj} & \mu_{jk} \\ -\mu_{ii} & -\mu_{ij} & \mu_{ik} \\ 0 & 00 & 0 \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} 0 & 0 & 0 \\ \eta_{ii} & \eta_{ji} & \eta_{ii} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ \mu_{ki} & \mu_{kj} & \mu_{kk} \\ -\mu_{ji} & -\mu_{jj} & -\mu_{jk} \end{bmatrix}\right\}\begin{bmatrix} n_i \\ n_j \\ n_k \end{bmatrix} \tag{94}$$

$$P_{ij}(i\neq j) = \varepsilon_{ijk}n_i(n_in_jn_k)\left\{\begin{bmatrix} \eta_{ii} & \eta_{ji} & \eta_{ki} \\ 0 & 0 & 0 \\ \eta_{ik} & \eta_{ik} & \eta_{kk} \end{bmatrix}\begin{bmatrix} \mu_{ki} & \mu_{ki} & \mu_{kk} \\ 0 & 0 & 0 \\ -\mu_{ii} & -\mu_{ij} & -\mu_{ik} \end{bmatrix} + \begin{bmatrix} \eta_{ij} & \eta_{jj} & \eta_{kk} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\mu_{ki} & -\mu_{ki} & -\mu_{kk} \\ 0 & 0 & 0 \\ \mu_{ii} & \mu_{ii} & \mu_{ik} \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} \eta_{ik} & \eta_{jk} & \eta_{kk} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mu_{ji} & \mu_{jj} & \mu_{jk} \\ -\mu_{ii} & -\mu_{ij} & -\mu_{ik} \\ 0 & 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} n_i \\ n_j \\ n_k \end{bmatrix} \tag{95}$$

$$Q_{ii}=n_i(n_i n_j n_k)\left\{\begin{bmatrix}0 & -\mu_{ik} & \mu_{ij}\\ 0 & -\mu_{jk} & \mu_{jj}\\ 0 & -\mu_{kk} & \mu_{kj}\end{bmatrix}\begin{bmatrix}0 & \zeta_{ji} & \zeta_{ki}\\ 0 & \zeta_{jj} & \zeta_{kj}\\ 0 & \zeta_{jk} & \zeta_{kk}\end{bmatrix}+\begin{bmatrix}-\mu_{ik} & 0 & \mu_{ii}\\ -\mu_{jk} & 0 & \mu_{ji}\\ -\mu_{kk} & 0 & \mu_{ki}\end{bmatrix}\begin{bmatrix}\zeta_{ji} & 0 & 0\\ \zeta_{jj} & 0 & 0\\ \zeta_{jk} & 0 & 0\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}\mu_{ij} & -\mu_{ii} & 0\\ \mu_{jj} & -\mu_{ji} & 0\\ \mu_{kj} & -\mu_{ki} & 0\end{bmatrix}\begin{bmatrix}\zeta_{ki} & 0 & 0\\ \zeta_{kj} & 0 & 0\\ \zeta_{kk} & 0 & 0\end{bmatrix}\right\}\begin{bmatrix}n_i\\ n_j\\ n_k\end{bmatrix} \tag{96}$$

$$Q_{ij}(i\neq j)=\varepsilon_{ijk}n_j(n_i n_j n_k)\left\{\begin{bmatrix}0 & -\mu_{ik} & \mu_{ij}\\ 0 & -\mu_{jk} & \mu_{jj}\\ 0 & -\mu_{kk} & \mu_{kj}\end{bmatrix}\begin{bmatrix}0 & \zeta_{ji} & \zeta_{ki}\\ 0 & \zeta_{jj} & \zeta_{kj}\\ 0 & \zeta_{jk} & \zeta_{kk}\end{bmatrix}+\begin{bmatrix}-\mu_{ik} & 0 & \mu_{ii}\\ -\mu_{jk} & 0 & \mu_{ji}\\ -\mu_{kk} & 0 & \mu_{ki}\end{bmatrix}\begin{bmatrix}\zeta_{ji} & 0 & 0\\ \zeta_{jj} & 0 & 0\\ \zeta_{jk} & 0 & 0\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}\mu_{ij} & -\mu_{ii} & 0\\ \mu_{ji} & -\mu_{ji} & 0\\ \mu_{kj} & -\mu_{ki} & 0\end{bmatrix}\begin{bmatrix}\zeta_{ki} & 0 & 0\\ \zeta_{kj} & 0 & 0\\ \zeta_{kk} & 0 & 0\end{bmatrix}\right\}\begin{bmatrix}n_i\\ n_j\\ n_k\end{bmatrix} \tag{97}$$

$$R_{ii}=(n_i n_j n_k)\left\{\begin{bmatrix}\mathscr{H}_{ii} & \mathscr{H}_{ji} & \mathscr{H}_{ki}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mu_{ii} & \mu_{ij} & \mu_{ik}\\ \mu_{ji} & \mu_{jj} & \mu_{jk}\\ \mu_{ki} & \mu_{kj} & \mu_{kk}\end{bmatrix}+\begin{bmatrix}\mu_{ii} & \mu_{ij} & \mu_{ik}\\ \mu_{ji} & \mu_{jj} & \mu_{ik}\\ \mu_{ki} & \mu_{ki} & \mu_{kk}\end{bmatrix}\begin{bmatrix}Z_{ii} & 0 & 0\\ Z_{ij} & 0 & 0\\ Z_{ik} & 0 & 0\end{bmatrix}\right.$$

$$+\begin{bmatrix}(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}\mu_{kk} & 0 & -\mu_{ki}\\ 0 & 0 & 0\\ -\mu_{ik} & 0 & \mu_{ii}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix} & (\eta_{ij}\eta_{jj}\eta_{ki})\begin{bmatrix}0 & -\mu_{kk} & -\mu_{kj}\\ 0 & 0 & 0\\ 0 & -\mu_{ik} & \mu_{ij}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix} & 0\\ (\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}0 & 0 & 0\\ \mu_{kk} & 0 & -\mu_{ki}\\ -\mu_{jk} & 0 & \mu_{ji}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix} & (\eta_{ij}\eta_{ji}\eta_{kj})\begin{bmatrix}0 & 0 & 0\\ 0 & \mu_{kk} & -\mu_{kj}\\ 0 & -\mu_{jk} & \mu_{jj}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix} & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$+\begin{bmatrix}(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}-\mu_{kj} & \mu_{ki} & 0\\ 0 & 0 & 0\\ \mu_{ij} & -\mu_{ii} & 0\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\ \zeta_{kj}\\ \zeta_{kk}\end{bmatrix} & 0 & (\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}0 & \mu_{kk} & -\mu_{ki}\\ 0 & - & 0\\ 0 & -\mu_{ik} & \mu_{ij}\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\ \zeta_{kj}\\ \zeta_{kk}\end{bmatrix}\\ (\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}0 & 0 & 0\\ -\mu_{kj} & \mu_{ki} & 0\\ \mu_{jj} & -\mu_{ji} & 0\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\ \zeta_{kj}\\ \zeta_{kk}\end{bmatrix} & 0 & (\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}0 & 0 & 0\\ 0 & \mu_{kk} & -\mu_{kj}\\ 0 & -\mu_{jk} & \mu_{jj}\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\ \zeta_{kj}\\ \zeta_{kk}\end{bmatrix}\\ 0 & 0 & 0\end{bmatrix}$$

$$+\begin{bmatrix}(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}-\mu_{jk} & 0 & \mu_{ji}\\ \mu_{ik} & 0 & -\mu_{ii}\\ 0 & 0\ 0 & 0\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix} & (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0 & -\mu_{jk} & \mu_{jj}\\ 0 & \mu_{ik} & -\mu_{ij}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix} & 0\\ 0 & 0 & 0\\ (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0 & 0\ 0 & 0\\ \mu_{kk} & 0 & -\mu_{ki}\\ -\mu_{jk} & 0 & \mu_{ji}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix} & (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0 & 0 & 0\\ 0 & \mu_{kk} & -\mu_{ki}\\ 0 & -\mu_{jk} & \mu_{jj}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix} & 0\end{bmatrix}$$

$$+\begin{vmatrix} (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} \mu_{jj} & -\mu_{ji} & 0 \\ -\mu_{ij} & \mu_{ii} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} & 0 & (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} 0 & -\mu_{jk} & \mu_{jj} \\ 0 & \mu_{ik} & -\mu_{ij} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} \\ 0 & 0 & 0 \\ (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} 0 & 0 & 0 \\ -\mu_{kj} & \mu_{ki} & 0 \\ \mu_{jj} & -\mu_{ji} & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} & 0 & (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_{kk} & -\mu_{kj} \\ 0 & -\mu_{ik} & \mu_{jj} \end{bmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} \end{vmatrix}\begin{bmatrix} n_i \\ n_j \\ n_k \end{bmatrix} \tag{98}$$

$$R_{ij}(i \neq j) = (n_i n_j n_k)\left\{\begin{vmatrix} \mathscr{H}_{ij} & \mathscr{H}_{jj} & \mathscr{H}_{kj} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}\begin{bmatrix} \mu_{ii} & \mu_{ij} & \mu_{ik} \\ \mu_{ji} & \mu_{jj} & \mu_{jk} \\ \mu_{ki} & \mu_{kj} & \mu_{kk} \end{bmatrix} + \begin{bmatrix} \mu_{ii} & \mu_{ij} & \mu_{ik} \\ \mu_{ji} & \mu_{jj} & \mu_{jk} \\ \mu_{ki} & \mu_{kj} & \mu_{kk} \end{bmatrix}\begin{bmatrix} 0 & Z_{ii} & 0 \\ 0 & Z_{ij} & 0 \\ 0 & Z_{ik} & 0 \end{bmatrix}\right.$$

$$+\begin{vmatrix} (\eta_{ii}\eta_{ji}\eta_{ki})\begin{vmatrix} -\mu_{kk} & 0 & \mu_{ki} \\ 0 & 0 & 0 \\ \mu_{ik} & 0 & -\mu_{ii} \end{vmatrix}\begin{bmatrix} \zeta_{ji} \\ \zeta_{jj} \\ \zeta_{jk} \end{bmatrix} & (\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} 0 & -\mu_{kk} & \mu_{kj} \\ 0 & 0 & 0 \\ 0 & \mu_{ik} & -\mu_{ij} \end{bmatrix}\begin{bmatrix} \zeta_{ji} \\ \zeta_{jj} \\ \zeta_{jk} \end{bmatrix} & 0 \\ (\eta_{ii}\eta_{ji}\eta_{ki})\begin{vmatrix} 0 & 0 & 0 \\ -\mu_{kk} & 0 & \mu_{ki} \\ \mu_{jk} & 0 & -\mu_{ji} \end{vmatrix}\begin{bmatrix} \zeta_{ji} \\ \zeta_{jj} \\ \zeta_{jk} \end{bmatrix} & (\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mu_{kk} & \mu_{kj} \\ 0 & \mu_{jk} & -\mu_{jj} \end{bmatrix}\begin{bmatrix} \zeta_{ji} \\ \zeta_{jj} \\ \zeta_{jk} \end{bmatrix} & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$+\begin{vmatrix} (\eta_{ii}\eta_{ji}\eta_{ki})\begin{vmatrix} \mu_{kj} & -\mu_{ki} & 0 \\ 0 & 0 & 0 \\ -\mu_{ij} & \mu_{ii} & 0 \end{vmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} & 0 & (\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} 0 & -\mu_{kk} & \mu_{kj} \\ 0 & 0 & 0 \\ 0 & \mu_{ik} & -\mu_{ij} \end{bmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} \\ (\eta_{ii}\eta_{ji}\eta_{ki})\begin{vmatrix} 0 & 0 & 0 \\ \mu_{kj} & -\mu_{ki} & 0 \\ -\mu_{jj} & \mu_{ji} & 0 \end{vmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} & 0 & (\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mu_{kk} & \mu_{kj} \\ 0 & \mu_{jk} & -\mu_{jj} \end{bmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} \\ 0 & 0 & 0 \end{vmatrix}$$

$$+\begin{vmatrix} 0 & 0 & 0 \\ (\eta_{ik}\eta_{jk}\eta_{kk})\begin{vmatrix} -\mu_{jk} & 0 & \mu_{ji} \\ \mu_{ik} & 0 & -\mu_{ii} \\ 0 & 0 & 0 \end{vmatrix}\begin{bmatrix} \zeta_{ji} \\ \zeta_{jj} \\ \zeta_{jk} \end{bmatrix} & (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} 0 & -\mu_{jk} & \mu_{jj} \\ 0 & \mu_{ik} & -\mu_{ij} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ji} \\ \zeta_{jj} \\ \zeta_{jk} \end{bmatrix} & 0 \\ (\eta_{ik}\eta_{jk}\eta_{kk})\begin{vmatrix} -\mu_{kk} & 0 & \mu_{ki} \\ 0 & 0 & 0 \\ \mu_{ik} & 0 & -\mu_{ii} \end{vmatrix}\begin{bmatrix} \zeta_{ji} \\ \zeta_{jj} \\ \zeta_{jk} \end{bmatrix} & (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} 0 & -\mu_{kk} & \mu_{kj} \\ 0 & 0 & 0 \\ 0 & \mu_{ik} & -\mu_{ij} \end{bmatrix}\begin{bmatrix} \zeta_{ji} \\ \zeta_{jj} \\ \zeta_{jk} \end{bmatrix} & 0 \end{vmatrix}$$

$$\left.+\begin{vmatrix} 0 & 0 & 0 \\ (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} \mu_{jj} & -\mu_{ji} & 0 \\ -\mu_{ij} & \mu_{ii} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} & 0 & (\eta_{ik}\eta_{jk}\eta_{kk})\begin{vmatrix} 0 & -\mu_{ik} & \mu_{jj} \\ 0 & \mu_{ik} & -\mu_{ij} \\ 0 & 0 & 0 \end{vmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} \\ (\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} \mu_{kj} & -\mu_{ki} & 0 \\ 0 & 0 & 0 \\ -\mu_{ij} & \mu_{ii} & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} & 0 & (\eta_{ik}\eta_{jk}\eta_{kk})\begin{vmatrix} 0 & -\mu_{kk} & \mu_{kj} \\ 0 & 0 & 0 \\ 0 & \mu_{ik} & -\mu_{ij} \end{vmatrix}\begin{bmatrix} \zeta_{ki} \\ \zeta_{kj} \\ \zeta_{kk} \end{bmatrix} \end{vmatrix}\right\}\begin{bmatrix} n_i \\ n_j \\ n_k \end{bmatrix} \tag{99}$$

$$S_{ii}=(n_i n_j n_k)\left\{\begin{bmatrix}0 & -M_{jj} & -M_{jk}\\ 0 & M_{ij} & M_{ik}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\varepsilon_{ij} & 0 & 0\\ \varepsilon_{jj} & 0 & 0\\ \varepsilon_{kj} & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & -M_{kj} & M_{kk}\\ 0 & 0 & 0\\ 0 & M_{ij} & M_{ik}\end{bmatrix}\begin{bmatrix}\varepsilon_{ik} & 0 & 0\\ \varepsilon_{jk} & 0 & 0\\ \varepsilon_{kk} & 0 & 0\end{bmatrix}\right.$$
$$\left.+\begin{bmatrix} & 0 & 0\\ \varepsilon_{ji} & \varepsilon_{jj} & \varepsilon_{jk}\\ \varepsilon_{kj} & \varepsilon_{kj} & \varepsilon_{kk}\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ M_{ji} & 0 & 0\\ M_{ki} & 0 & 0\end{bmatrix}-M_{ii}\begin{bmatrix}0 & 0 & 0\\ 0 & \varepsilon_{jj} & \varepsilon_{jk}\\ 0 & \varepsilon_{kj} & \varepsilon_{kk}\end{bmatrix}\right\}\begin{bmatrix}n_i\\ n_j\\ n_k\end{bmatrix}$$

$$S_{ij}\,(i\neq j)=(n_i n_j n_{12})\left\{\begin{bmatrix}0 & M_{jj} & M_{jk}\\ 0 & -M_{ij} & -M_{ik}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\varepsilon_{ii} & 0 & 0\\ \varepsilon_{ji} & 0 & 0\\ \varepsilon_{kj} & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & -M_{kj} & -M_{1212}\\ 0 & M_{jj} & M_{j12}\end{bmatrix}\begin{bmatrix}\varepsilon_{ik} & 0 & 0\\ \varepsilon_{jk} & 0 & 0\\ \varepsilon_{kk} & 0 & 0\end{bmatrix}\right.$$
$$\left.+\begin{bmatrix}0 & 0 & 0\\ \varepsilon_{ji} & \varepsilon_{jj} & \varepsilon_{jk}\\ \varepsilon_{ki} & \varepsilon_{kj} & \varepsilon_{kk}\end{bmatrix}\begin{bmatrix}0 & M_{ii} & 0\\ 0 & 0 & 0\\ 0 & M_{ki} & 0\end{bmatrix}-M_{ji}\begin{bmatrix}0 & 0 & 0\\ \varepsilon_{ji} & 0 & \varepsilon_{jk}\\ \varepsilon_{ki} & 0 & \varepsilon_{kk}\end{bmatrix}\right\}\begin{bmatrix}n_i\\ n_j\\ n_k\end{bmatrix} \tag{100}$$

$$U_{ij}=n_i\left\{(\mathscr{H}_{ii}\mathscr{H}_{ji}\mathscr{H}_{ki})\begin{bmatrix}-\mu_{ik} & 0 & M_{ii}\\ -\mu_{jk} & 0 & M_{ji}\\ -\mu_{kk} & 0 & M_{kk}\end{bmatrix}\begin{bmatrix}\xi_{ji}\\ \xi_{jj}\\ \xi_{jk}\end{bmatrix}+(H_{ii}H_{ji}H_{ki})\begin{bmatrix}\mu_{ij} & -\mu_{11} & 0\\ \mu_{jj} & -\mu_{ji} & 0\\ \mu_{kj} & -\mu_{ki} & 0\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\ \zeta_{kj}\\ \zeta_{kk}\end{bmatrix}\right.$$
$$\left.+(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}-\mu_{ki} & -\mu_{kjc} & -\mu_{kk}\\ 0 & 0 & 0\\ \mu_{ii} & \mu_{ij} & \mu_{ik}\end{bmatrix}\begin{bmatrix}Z_{ii}\\ Z_{ij}\\ Z_{ik}\end{bmatrix}+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}\mu_{ji} & \mu_{jj} & \mu_{jk}\\ -\mu_{ii} & -\mu_{ij} & -\mu_{ik}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}Z_{ii}\\ Z_{ij}\\ Z_{ik}\end{bmatrix}\right\}$$
$$+n_j\left\{(\mathscr{H}_{ii}\mathscr{H}_{ji}\mathscr{H}_{ki})\begin{bmatrix}0 & -\mu_{ik} & \mu_{ij}\\ 0 & -\mu_{jk} & \mu_{jj}\\ 0 & -\mu_{kk} & \mu_{kj}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix}+(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}0 & 0 & 0\\ -\mu_{ki} & -\mu_{kj} & -\mu_{kk}\\ \mu_{ji} & \mu_{jj} & \mu_{jk}\end{bmatrix}\begin{bmatrix}Z_{ii}\\ Z_{ij}\\ Z_{ik}\end{bmatrix}\right\}$$
$$+n_k\left\{(\mathscr{H}_{ii}\mathscr{H}_{ji}\mathscr{H}_{ki})\begin{bmatrix}0 & -\mu_{ik} & \mu_{ij}\\ 0 & -\mu_{jk} & \mu_{jj}\\ 0 & -\mu_{kk} & \mu_{ij}\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\ \zeta_{kj}\\ \zeta_{kk}\end{bmatrix}+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0 & 0 & 0\\ -\mu_{ki} & -\mu_{kj} & -\mu_{kk}\\ \mu_{ji} & \mu_{jj} & \mu_{jk}\end{bmatrix}\begin{bmatrix}Z_{ii}\\ Z_{ij}\\ Z_{ik}\end{bmatrix}\right\} \tag{101}$$

$$U_{ii}\,(i\neq j)=\varepsilon_{ijk}\left[n_i\left\{(\mathscr{H}_{ij}\mathscr{H}_{jj}\mathscr{H}_{ik})\begin{bmatrix}-\mu_{ik} & 0 & \mu_{ii}\\ -\mu_{jk} & 0 & \mu_{ji}\\ -\mu_{kk} & 0 & \mu_{ii}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix}+(\mathscr{H}_{ij}\mathscr{H}_{jj}\mathscr{H}_{kj})\begin{bmatrix}\mu_{ij} & 0\\ \mu_{jj} & 0\\ \mu_{kj} & 0\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\ \zeta_{kj}\\ \zeta_{kk}\end{bmatrix}\right.\right.$$
$$\left.+(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix}\mu_{ki} & \mu_{kj} & \mu_{kk}\\ 0 & 0 & 0\\ -\mu_{ii} & -\mu_{ij} & -\mu_{ik}\end{bmatrix}\begin{bmatrix}Z_{ii}\\ Z_{ij}\\ Z_{ik}\end{bmatrix}\right\}+n_j\left\{(\mathscr{H}_{ij}\mathscr{H}_{jj}\mathscr{H}_{kj})\begin{bmatrix}0 & -\mu_{ij}\\ 0 & -\mu_{jj}\\ 0 & -\mu_{kj}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\ \zeta_{jj}\\ \zeta_{jk}\end{bmatrix}\right.$$
$$\left.+(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix}0 & 0 & 0\\ \mu_{ki} & \mu_{kj} & \mu_{kk}\\ -\mu_{ji} & -\mu_{jj} & -\mu_{jk}\end{bmatrix}\begin{bmatrix}Z_{ii}\\ Z_{ij}\\ Z_{ik}\end{bmatrix}+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}\mu_{ji} & \mu_{jj} & \mu_{ji2}\\ -\mu_{ii} & -\mu_{ii} & -\mu_{ii2}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}Z_{ii}\\ Z_{ij}\\ Z_{ik}\end{bmatrix}\right\}$$
$$\left.+n_{i2}\left\{(\mathscr{H}_{ij}\mathscr{H}_{jj}\mathscr{H}_{kj})\begin{bmatrix}0 & -\mu_{ik} & \mu_{ij}\\ 0 & -\mu_{jk} & \mu_{jj}\\ 0 & -\mu_{kk} & \mu_{kj}\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\ \zeta_{kj}\\ \zeta_{kk}\end{bmatrix}+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}\mu_{ki} & \mu_{kj} & \mu_{kk}\\ 0 & 0 & 0\\ -\mu_{ii} & -\mu_{ij} & -\mu_{ik}\end{bmatrix}\begin{bmatrix}Z_{ii}\\ Z_{ij}\\ Z_{ik}\end{bmatrix}\right\}\right] \tag{102}$$

$$V_{ii}=n_i\left\{-(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}0 & M_{ij} & M_{ik}\\ 0 & M_{jj} & M_{jk}\\ 0 & M_{kj} & M_{kk}\end{bmatrix}\begin{bmatrix}\varepsilon_{ik}\\ \varepsilon_{jk}\\ \varepsilon_{kk}\end{bmatrix}+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0 & M_{ij} & M_{ik}\\ 0 & M_{jj} & M_{jk}\\ 0 & M_{kj} & M_{kk}\end{bmatrix}\begin{bmatrix}\varepsilon_{ij}\\ \varepsilon_{jj}\\ \varepsilon_{kj}\end{bmatrix}\right.$$

$$
\begin{aligned}
&-(\varepsilon_{ki}\varepsilon_{kj}\varepsilon_{kk})\begin{bmatrix}0&0&0\\M_{ji}&M_{jj}&M_{jk}\\M_{ki}&M_{kj}&M_{kk}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\\zeta_{jj}\\\zeta_{jk}\end{bmatrix}+(\varepsilon_{ji}\varepsilon_{jj}\varepsilon_{jk})\begin{bmatrix}0&0&0\\M_{ji}&M_{jj}&M_{jk}\\M_{ki}&M_{kj}&M_{kk}\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\\zeta_{kj}\\\zeta_{kk}\end{bmatrix}\Bigg]\\
&+n_j\Bigg\{(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}0&M_{ii}&0\\0&M_{ji}&0\\0&M_{ki}&0\end{bmatrix}\begin{bmatrix}\varepsilon_{ik}\\\varepsilon_{jk}\\\varepsilon_{kk}\end{bmatrix}-(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0&M_{ii}&0\\0&M_{ji}&0\\0&M_{ki}&0\end{bmatrix}\begin{bmatrix}\varepsilon_{ij}\\\varepsilon_{jj}\\\varepsilon_{kj}\end{bmatrix}\\
&+(\varepsilon_{ki}\varepsilon_{kj}\varepsilon_{kk})\begin{bmatrix}0&0&0\\M_{ii}&M_{ij}&M_{ik}\\0&0&0\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\\zeta_{jj}\\\zeta_{jk}\end{bmatrix}-(\varepsilon_{ji}\varepsilon_{jj}\varepsilon_{jk})\begin{bmatrix}0&0&0\\M_{ii}&M_{ij}&M_{ik}\\0&0&0\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\\zeta_{kj}\\\zeta_{kk}\end{bmatrix}\Bigg\}\\
&+n_k\Bigg\{(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}0&0&M_{ii}\\0&0&M_{ji}\\0&0&M_{ki}\end{bmatrix}\begin{bmatrix}\varepsilon_{ik}\\\varepsilon_{jk}\\\varepsilon_{kk}\end{bmatrix}-(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0&0&M_{ii}\\0&0&M_{ji}\\0&0&M_{ki}\end{bmatrix}\begin{bmatrix}\varepsilon_{ij}\\\varepsilon_{jj}\\\varepsilon_{kj}\end{bmatrix}\\
&+(\varepsilon_{ki}\varepsilon_{kj}\varepsilon_{kk})\begin{bmatrix}0&0&0\\0&0&0\\M_{ii}&M_{ij}&M_{ik}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\\zeta_{jj}\\\zeta_{jk}\end{bmatrix}-(\varepsilon_{ji}\varepsilon_{jj}\varepsilon_{jk})\begin{bmatrix}0&0&0\\0&0&0\\M_{ii}&M_{ij}&M_{ik}\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\\zeta_{kj}\\\zeta_{kk}\end{bmatrix}\Bigg\}\Bigg]
\end{aligned}
\tag{103}
$$

$$
\begin{aligned}
V_{ij}(i\neq j)=\varepsilon_{ijk}\Bigg[&n_i\Bigg\{(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix}0&M_{ij}&M_{ik}\\0&M_{jj}&M_{jk}\\0&M_{kj}&M_{kk}\end{bmatrix}\begin{bmatrix}\varepsilon_{ik}\\\varepsilon_{jk}\\\varepsilon_{kk}\end{bmatrix}-(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0&M_{ij}&M_{ik}\\0&M_{jj}&M_{jk}\\0&M_{kj}&M_{kk}\end{bmatrix}\begin{bmatrix}\varepsilon_{ii}\\\varepsilon_{ji}\\\varepsilon_{ki}\end{bmatrix}\\
&+(\varepsilon_{ki}\varepsilon_{kj}\varepsilon_{kk})\begin{bmatrix}M_{ji}&M_{jj}&M_{jk}\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\\zeta_{jj}\\\zeta_{jk}\end{bmatrix}-(\varepsilon_{ji}\varepsilon_{jj}\varepsilon_{jk})\begin{bmatrix}M_{ji}&M_{jj}&M_{jk}\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\\zeta_{kj}\\\zeta_{kk}\end{bmatrix}\Bigg\}\\
&+n_j\Bigg\{-(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix}0&M_{ii}&0\\0&M_{ji}&0\\0&M_{ki}&0\end{bmatrix}\begin{bmatrix}\varepsilon_{ik}\\\varepsilon_{jk}\\\varepsilon_{kk}\end{bmatrix}+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0&M_{ii}&0\\0&M_{ji}&0\\0&M_{ki}&0\end{bmatrix}\begin{bmatrix}\varepsilon_{ii}\\\varepsilon_{ji}\\\varepsilon_{ki}\end{bmatrix}\\
&-(\varepsilon_{ki}\varepsilon_{kj}\varepsilon_{kk})\begin{bmatrix}M_{ii}&M_{ij}&M_{ik}\\0&0&0\\M_{ki}&M_{kj}&M_{kk}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\\zeta_{jj}\\\zeta_{jk}\end{bmatrix}+(\varepsilon_{ji}\varepsilon_{jj}\varepsilon_{jk})\begin{bmatrix}M_{ik}&M_{ij}&M_{ik}\\0&0&0\\M_{ki}&M_{kj}&M_{kk}\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\\zeta_{kj}\\\zeta_{kk}\end{bmatrix}\Bigg\}\\
&+n_k\Bigg\{-(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix}0&0&M_{ii}\\0&0&M_{ji}\\0&0&M_{ki}\end{bmatrix}\begin{bmatrix}\varepsilon_{ik}\\\varepsilon_{jk}\\\varepsilon_{kk}\end{bmatrix}+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}0&0&M_{ii}\\0&0&M_{ji}\\0&0&M_{ki}\end{bmatrix}\begin{bmatrix}\varepsilon_{ii}\\\varepsilon_{ji}\\\varepsilon_{ki}\end{bmatrix}\\
&+(\varepsilon_{ki}\varepsilon_{kj}\varepsilon_{kk})\begin{bmatrix}0&0&0\\0&0&0\\M_{ji}&M_{jj}&M_{jk}\end{bmatrix}\begin{bmatrix}\zeta_{ji}\\\zeta_{jj}\\\zeta_{jk}\end{bmatrix}-(\varepsilon_{ji}\varepsilon_{jj}\varepsilon_{jk})\begin{bmatrix}0&0&0\\0&0&0\\M_{ji}&M_{jj}&M_{jk}\end{bmatrix}\begin{bmatrix}\zeta_{ki}\\\zeta_{kj}\\\zeta_{kk}\end{bmatrix}\Bigg\}\Bigg]
\end{aligned}
\tag{104}
$$

$$
\left.
\begin{aligned}
T_{ii}=&(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}M_{ii}&M_{ij}&M_{ik}\\M_{ji}&M_{jj}&M_{jk}\\M_{ki}&M_{kj}&M_{kk}\end{bmatrix}\begin{bmatrix}0&\zeta_{ki}&-\zeta_{ji}\\0&\zeta_{kj}&-\zeta_{jj}\\0&\zeta_{kk}&-\zeta_{jk}\end{bmatrix}\begin{bmatrix}\varepsilon_{ik}\\\varepsilon_{jk}\\\varepsilon_{kk}\end{bmatrix}\\
&+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix}M_{ii}&M_{ij}&M_{ik}\\M_{ji}&M_{jj}&M_{jk}\\M_{ki}&M_{kj}&M_{kk}\end{bmatrix}\begin{bmatrix}0&-\zeta_{ki}&\zeta_{ji}\\0&-\zeta_{kj}&\zeta_{jj}\\0&-\zeta_{kk}&\zeta_{jk}\end{bmatrix}\begin{bmatrix}\varepsilon_{ij}\\\varepsilon_{jj}\\\varepsilon_{kj}\end{bmatrix}\\
T_{ij}(i\neq j)=&(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix}M_{ii}&M_{ij}&M_{ik}\\M_{ji}&M_{jj}&M_{jk}\\M_{ki}&M_{kj}&M_{kk}\end{bmatrix}\begin{bmatrix}0&-\zeta_{ki}&\zeta_{ji}\\0&-\zeta_{kj}&\zeta_{jj}\\0&-\zeta_{kk}&\zeta_{jk}\end{bmatrix}\begin{bmatrix}\varepsilon_{ik}\\\varepsilon_{jk}\\\varepsilon_{kk}\end{bmatrix}
\end{aligned}
\right|
\tag{105}
$$

$$+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} M_{ii} & M_{ij} & M_{ik} \\ M_{ji} & M_{jj} & M_{jk} \\ M_{ki} & M_{kj} & M_{kk} \end{bmatrix}\begin{bmatrix} 0 & \zeta_{ki} & -\zeta_{ji} \\ 0 & \zeta_{kj} & -\zeta_{jj} \\ 0 & \zeta_{kk} & -\zeta_{jk} \end{bmatrix}\begin{bmatrix} \varepsilon_{ii} \\ \varepsilon_{ji} \\ \varepsilon_{ki} \end{bmatrix}$$

한편 行列式을 예컨대 第 1行에 따라 展開하면

$$\det(\underline{\underline{W}}_E) = W_{E11}\,\mathrm{cof}(W_{E11}) + W_{E12}\,\mathrm{cof}(W_{E12}) + W_{E13}\,\mathrm{cof}(W_{E13}) \tag{106}$$

波動行列 W_{E11}, W_{E12}, W_{E13}는 모두 k의 二次式이며 그들의 餘因子는 모두 k의 四次式이므로 波動演算子行列의 行列式은 k의 六次式이다. 그러나 긴 計算끝에 最高次의 2項이 消滅되어 式(106)의 行列式은 결국 다음과 같은 k의 四次式이 됨을 알 수 있다.

$$\begin{aligned}\det(\underline{\underline{W}}_E) = {} & \frac{k^4}{(\det(\underline{\underline{\mu}}))^2}\left\{\frac{\omega^2}{\det(\underline{\underline{\mu}})}\Gamma + k_0^2\Delta\right\} + \frac{k^3\omega}{(\det(\underline{\underline{\mu}}))^2}\left\{\frac{\omega^2}{\det(\underline{\underline{\mu}})}\Theta + k_0^2\Lambda\right\} \\ & + \frac{k^2}{\det(\underline{\underline{\mu}})}\left\{\frac{\omega^4}{(\det(\underline{\underline{\mu}}))^2}\Xi + \frac{\omega^2 k_i^2}{\det(\underline{\underline{\mu}})}\,\mathrm{O} + k_0^4\Pi\right\} \\ & + \frac{k\,\omega}{\det(\underline{\underline{\mu}})}\left\{\frac{\omega^4}{(\det(\underline{\underline{\mu}}))^2}\Sigma + \frac{\omega^2 k_0^2}{\det(\underline{\underline{\mu}})}\Upsilon + k_0^4\Phi\right\} \\ & + \frac{\omega^2}{\det(\underline{\underline{\mu}})}\left\{\frac{\omega^4}{(\det(\underline{\underline{\mu}}))^2}\mathrm{X} + \frac{\omega^2 k_0^2}{\det(\underline{\underline{\mu}})}\Psi + k_0^4\Omega\right\} + k_0^6\det(\underline{\underline{\varepsilon}})\end{aligned} \tag{107}$$

여기서 Γ, Δ, Θ, Λ, Ξ, O, Π, Σ, Υ, Φ, X, Ψ, 및 Ω는 다음과 같다.

$$\begin{aligned}\Gamma = {} & (n_1N_{11}+n_2N_{12}+n_3N_{13})\,n_1\,\underline{n}^T(\mathcal{M})\underline{n} + (P_{11}+Q_{11})\{(n_2K_{31}-n_3K_{21})+(n_2L_{13}-n_3L_{12})\} \\ & + (P_{12}+Q_{11})\{(n_2K_{32}-n_3K_{22})+(n_3L_{11}-n_1L_{13})\} + (P_{13}+Q_{13})\{(n_2K_{33}-n_3K_{23})+(n_1L_{12}-n_2L_{11})\} \\ & + R_{11}J_{11}+R_{12}J_{12}+R_{13}J_{13}\end{aligned} \tag{108}$$

$$\begin{aligned}\Delta = \underline{n}^T \Bigg\{ & \underline{\underline{\varepsilon}} + (n_1^2\varepsilon_{11}+n_2^2\varepsilon_{22}+n_3^2\varepsilon_{33})\begin{bmatrix} 0 & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & 0 & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & 0 \end{bmatrix} \\ & + \begin{bmatrix} n_3^2(\varepsilon_{13}+\varepsilon_{31})(M_{13}+M_{31}) & n_3^2\left\{\begin{array}{l}(\varepsilon_{13}+\varepsilon_{31})(\mathcal{M}_{23}+\mathcal{M}_{23}) \\ +(\varepsilon_{23}+\varepsilon_{32})(\mathcal{M}_{13}+\mathcal{M}_{31})\end{array}\right\} & n_2^2\left\{\begin{array}{l}(\varepsilon_{12}+\varepsilon_{21})(\mathcal{M}_{23}+\mathcal{M}_{32}) \\ +(\varepsilon_{23}+\varepsilon_{32})(\mathcal{M}_{12}+\mathcal{M}_{21})\end{array}\right\} \\ & n_1^2(\varepsilon_{12}+\varepsilon_{21})(\mathcal{M}_{12}+\mathcal{M}_{21}) & n_1^2\left\{\begin{array}{l}(\varepsilon_{12}+\varepsilon_{21})(\mathcal{M}_{13}+\mathcal{M}_{31}) + \\ (\varepsilon_{13}+\varepsilon_{31})(\mathcal{M}_{12}+\mathcal{M}_{21})\end{array}\right\} \\ & & n_2^2(\varepsilon_{23}+\varepsilon_{32})(\mathcal{M}_{23}+\mathcal{M}_{32}) \end{bmatrix}\Bigg\}\underline{n}\end{aligned} \tag{109}$$

$$\begin{aligned}\Theta = {} & -(P_{11}+Q_{11})N_{11} - (P_{12}+Q_{12})N_{12} - (P_{13})N_{13} + R_{11}\{(n_2K_{31}-n_3K_{21})+(n_2L_{13}-n_3L_{12})\} \\ & + R_{12}\{(n_2K_{32}-n_3K_{22})+(n_3L_{11}-n_1L_{13})\} + R_{13}\{(n_2K_{33}-n_3K_{23})+(n_1L_{12}-n_2L_{11})\} \\ & + U_{11}J_{11}+U_{12}J_{12}+U_{13}J_{13}\end{aligned} \tag{110}$$

$$\Lambda = \underline{n}^T\left\{ n_1 \begin{bmatrix} (\mathcal{M}_{21}\ -\mathcal{M}_{11}\ 0)\begin{bmatrix} \eta_{13} & 0 & -\eta_{11} \\ \eta_{23} & 0 & -\eta_{21} \\ \eta_{33} & 0 & -\eta_{31} \end{bmatrix} + (\mathcal{M}_{31}\ 0\ -\mathcal{M}_{11})\begin{bmatrix} -\eta_{12} & \eta_{11} & 0 \\ -\eta_{22} & \eta_{21} & 0 \\ -\eta_{32} & \eta_{31} & 0 \end{bmatrix} \\ (\mathcal{M}_{21}\ -\mathcal{M}_{11}\ 0)\begin{bmatrix} 0 & \eta_{13} & -\eta_{12} \\ 0 & \eta_{23} & -\eta_{22} \\ 0 & \eta_{33} & -\eta_{32} \end{bmatrix} + (0\ \mathcal{M}_{31}\ -\mathcal{M}_{21})\begin{bmatrix} -\eta_{12} & \eta_{11} & 0 \\ -\eta_{22} & \eta_{21} & 0 \\ -\eta_{32} & \eta_{31} & 0 \end{bmatrix} \\ (\mathcal{M}_{31}\ 0\ -\mathcal{M}_{11})\begin{bmatrix} 0 & \eta_{13} & -\eta_{12} \\ 0 & \eta_{23} & -\eta_{22} \\ 0 & \eta_{33} & -\eta_{32} \end{bmatrix} + (0\ \mathcal{M}_{31}\ -\mathcal{M}_{21})\begin{bmatrix} -\eta_{13} & 0 & \eta_{11} \\ -\eta_{23} & 0 & \eta_{21} \\ -\eta_{33} & 0 & \eta_{31} \end{bmatrix} \end{bmatrix} \right.$$

$$
\left[(\mathscr{M}_{22}-\mathscr{M}_{12}\ 0)\begin{bmatrix}\eta_{13}&0&-\eta_{11}\\\eta_{23}&0&-\eta_{21}\\\eta_{33}&0&-\eta_{31}\end{bmatrix}+(\mathscr{M}_{32}\ 0\ -\mathscr{M}_{12})\begin{bmatrix}-\eta_{12}&\eta_{11}&0\\-\eta_{22}&\eta_{21}&0\\-\eta_{32}&\eta_{31}&0\end{bmatrix}\right]
$$

$$
+n^2\left[(\mathscr{M}_{22}-\mathscr{M}_{12}\ 0)\begin{bmatrix}0&\eta_{13}&-\eta_{12}\\0&\eta_{23}&-\eta_{22}\\0&\eta_{33}&-\eta_{32}\end{bmatrix}+(0\ \mathscr{M}_{32}-\mathscr{M}_{22})\begin{bmatrix}-\eta_{12}&\eta_{11}&0\\-\eta_{22}&\eta_{21}&0\\-\eta_{32}&\eta_{31}&0\end{bmatrix}\right.
$$

$$
\left.(\mathscr{M}_{32}\ 0-\mathscr{M}_{12})\begin{bmatrix}0&\eta_{13}&-\eta_{12}\\0&\eta_{23}&-\eta_{32}\\0&\eta_{33}&-\eta_{32}\end{bmatrix}+(0\ \mathscr{M}_{32}-\mathscr{M}_{22})\begin{bmatrix}-\eta_{13}&0&\eta_{11}\\-\eta_{23}&0&\eta_{11}\\-\eta_{33}&0&\eta_{31}\end{bmatrix}\right]
$$

$$
\left[(\mathscr{M}_{23}-\mathscr{M}_{13}\ 0)\begin{bmatrix}\eta_{13}&0&-\eta_{11}\\\eta_{23}&0&-\eta_{21}\\\eta_{33}&0&-\eta_{31}\end{bmatrix}+(\mathscr{M}_{33}\ 0-\mathscr{M}_{13})\begin{bmatrix}-\eta_{12}&0&\eta_{11}\\-\eta_{22}&0&\eta_{21}\\-\eta_{32}&0&\eta_{31}\end{bmatrix}\right.
$$

$$
+n_3\left[(\mathscr{M}_{23}-\mathscr{M}_{13}\ 0)\begin{bmatrix}0&\eta_{13}&-\eta_{12}\\0&\eta_{23}&-\eta_{22}\\0&\eta_{33}&-\eta_{32}\end{bmatrix}+(0\ \mathscr{M}_{33}-\mathscr{M}_{23})\begin{bmatrix}-\eta_{12}&0&\eta_{11}\\-\eta_{22}&0&\eta_{21}\\-\eta_{32}&0&\eta_{31}\end{bmatrix}\right]\begin{bmatrix}\varepsilon_{11}&\varepsilon_{21}&\varepsilon_{31}\\\varepsilon_{12}&\varepsilon_{22}&\varepsilon_{32}\\\varepsilon_{13}&\varepsilon_{23}&\varepsilon_{33}\end{bmatrix}\underline{n}
$$

$$
\left.(\mathscr{M}_{33}\ 0-\mathscr{M}_{13})\begin{bmatrix}0&\eta_{13}&-\eta_{12}\\0&\eta_{23}&-\eta_{22}\\0&\eta_{33}&-\eta_{32}\end{bmatrix}+(0\ \mathscr{M}_{33}-\mathscr{M}_{23})\begin{bmatrix}-\eta_{13}&\eta_{11}&0\\-\eta_{23}&\eta_{11}&0\\-\eta_{33}&\eta_{31}&0\end{bmatrix}\right]\Bigg]
$$

$$
+n_1^3(\varepsilon_{11}\varepsilon_{21}\varepsilon_{31})\left\{\begin{bmatrix}-\zeta_{21}&-\zeta_{22}&-\zeta_{23}\\\zeta_{11}&\zeta_{12}&\zeta_{13}\\0&0&0\end{bmatrix}\begin{bmatrix}\mathscr{M}_{13}\\0\\\mathscr{M}_{11}\end{bmatrix}+\begin{bmatrix}\zeta_{31}&\zeta_{32}&\zeta_{33}\\0&0&0\\-\zeta_{11}&-\zeta_{12}&-\zeta_{13}\end{bmatrix}\begin{bmatrix}\mathscr{M}_{12}\\-\mathscr{M}_{11}\\0\end{bmatrix}\right\}
$$

$$
+n_2^3(\varepsilon_{12}\varepsilon_{22}\varepsilon_{32})\left\{\begin{bmatrix}+\zeta_{11}&-\zeta_{12}&-\zeta_{23}\\\zeta_{11}&\zeta_{12}&\zeta_{13}\\0&0&\end{bmatrix}\begin{bmatrix}0\\\mathscr{M}_{23}\\-\mathscr{M}_{22}\end{bmatrix}+\begin{bmatrix}0&0&0\\\zeta_{31}&\zeta_{32}&\zeta_{33}\\-\zeta_{21}&-\zeta_{22}&-\zeta_{23}\end{bmatrix}\begin{bmatrix}\mathscr{M}_{22}\\-\mathscr{M}_{21}\\0\end{bmatrix}\right\}
$$

$$
+n_3^3(\varepsilon_{13}\varepsilon_{23}\varepsilon_{33}\left\{\begin{bmatrix}-\zeta_{31}&-\zeta_{32}&-\zeta_{23}\\0&0&\zeta_{13}\\\zeta_{11}&\zeta_{12}&0\end{bmatrix}\begin{bmatrix}0\\\mathscr{M}_{33}\\-\mathscr{M}_{32}\end{bmatrix}+\begin{bmatrix}0&0&0\\\zeta_{31}&\zeta_{32}&\zeta_{33}\\-\zeta_{21}&-\zeta_{22}&-\zeta_{23}\end{bmatrix}\begin{bmatrix}\mathscr{M}_{33}\\0\\-\mathscr{M}_{31}\end{bmatrix}\right\}
$$

$$
+n_1^2n_2(\varepsilon_{11}\varepsilon_{21}\varepsilon_{31})\left\{\begin{bmatrix}-\zeta_{21}&-\zeta_{22}&-\zeta_{23}\\\zeta_{11}&\zeta_{12}&\zeta_{13}\\0&0&0\end{bmatrix}\left(\begin{bmatrix}\mathscr{M}_{23}\\0\\-\mathscr{M}_{21}\end{bmatrix}+\begin{bmatrix}0\\\mathscr{M}_{13}\\-\mathscr{M}_{12}\end{bmatrix}\right)+\begin{bmatrix}\zeta_{21}&\zeta_{22}&\zeta_{23}\\0&0&0\\-\zeta_{11}&-\zeta_{12}&-\zeta_{13}\end{bmatrix}\begin{bmatrix}\mathscr{M}_{22}\\-\mathscr{M}_{12}\\0\end{bmatrix}\right\}
$$

$$
+n_1n_2^2(\varepsilon_{12}\varepsilon_{22}\varepsilon_{32})\left\{\begin{bmatrix}-\zeta_{21}&-\zeta_{22}&-\zeta_{13}\\\zeta_{11}&\zeta_{12}&0\\0&0&\end{bmatrix}\left(\begin{bmatrix}\mathscr{M}_{23}\\0\\-\mathscr{M}_{21}\end{bmatrix}+\begin{bmatrix}0\\\mathscr{M}_{13}\\-\mathscr{M}_{12}\end{bmatrix}\right)+\begin{bmatrix}0&0&0\\\zeta_{31}&\zeta_{32}&\zeta_{33}\\-\zeta_{21}&-\zeta_{22}&-\zeta_{23}\end{bmatrix}\begin{bmatrix}\mathscr{M}_{12}\\-\mathscr{M}_{11}\\0\end{bmatrix}\right\} \quad (111)
$$

$$
\Xi=-R_{11}U_{11}-R_{12}U_{12}-R_{13}U_{13}+U_{11}\{(n_2K_{31}-n_3K_{21})+(n_2L_{13}-n_3L_{12})\}+U_{12}\{(n_2K_{32}-n_3K_{22})+(n_3L_{11}-n_1L_{13})\}
$$

$$
+U_{13}\{(n_2K_{33}-n_3K_{23})+(n_1L_{12}-n_2L_{11})\}+(\mathscr{H}_{11}\mathscr{H}_{21}\mathscr{H}_{31})(\mathscr{M})(Z_{11}\ Z_{12}\ Z_{13})^TJ_{11}+(\mathscr{H}_{12}\mathscr{H}_{22}\mathscr{H}_{32})(\mathscr{M})(Z_{11}\ Z_{12}\ Z_{13})^TJ_{12}
$$

$$
+(\mathscr{H}_{13}\mathscr{H}_{23}\mathscr{H}_{33})(\mathscr{M})(Z_{11}\ Z_{12}\ Z_{13})^TJ_{13} \quad (112)
$$

$$
0=\underline{n}^T\underline{\underline{g}}\ \underline{n} \quad (113)
$$

$$
g_{ii}=(\mathscr{M}_{1i}\mathscr{M}_{2i}\mathscr{M}_{3i})(\mathscr{H})(\varepsilon_{i1}\ \varepsilon_{i2}\ \varepsilon_{i3})^T+(\varepsilon_{1i}\varepsilon_{2i}\ \varepsilon_{3i})(Z)(\mathscr{M}_{i1}\mathscr{M}_{i2}\mathscr{M}_{i3})^T
$$

$$
+(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix}\mathscr{M}_{kk}&0&-\mathscr{M}_{ki}\\0&0&0\\-\mathscr{M}_{ik}&0&\mathscr{M}_{ii}\end{bmatrix}\begin{bmatrix}-\zeta_{ji}&\zeta_{ii}&0\\-\zeta_{jj}&\zeta_{ij}&0\\-\zeta_{jk}&\zeta_{jk}&0\end{bmatrix}\begin{bmatrix}\varepsilon_{ij}\\\varepsilon_{jj}\\\varepsilon_{kj}\end{bmatrix}
$$

$$
+(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix}\mathscr{M}_{kk}&0&-\mathscr{M}_{ki}\\0&0&0\\-\mathscr{M}_{ik}&0&\mathscr{M}_{ii}\end{bmatrix}\begin{bmatrix}\zeta_{ji}&-\zeta_{ii}&0\\\zeta_{jj}&-\zeta_{ij}&0\\\zeta_{jk}&-\zeta_{ik}&0\end{bmatrix}\begin{bmatrix}\varepsilon_{ii}\\\varepsilon_{ji}\\\varepsilon_{ki}\end{bmatrix}
$$

$$
\begin{aligned}
&+(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} -\mathscr{M}_{kj} & -\mathscr{M}_{ki} & 0 \\ 0 & 0 & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ki} & 0 & -\zeta_{ii} \\ \zeta_{kj} & 0 & -\zeta_{ij} \\ \zeta_{kk} & 0 & -\zeta_{ik} \end{bmatrix}\begin{bmatrix} \varepsilon_{ij} \\ \varepsilon_{jj} \\ \varepsilon_{kj} \end{bmatrix} \\
&+(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix} \mathscr{M}_{kj} & -\mathscr{M}_{ki} & 0 \\ 0 & 0 & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \end{bmatrix}\begin{bmatrix} -\zeta_{ki} & 0 & \zeta_{ii} \\ -\zeta_{kj} & 0 & \zeta_{ij} \\ -\zeta_{kk} & 0 & \zeta_{ik} \end{bmatrix}\begin{bmatrix} \varepsilon_{ii} \\ \varepsilon_{ji} \\ \varepsilon_{ki} \end{bmatrix} \\
&+(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} \mathscr{M}_{jk} & 0 & \mathscr{M}_{ji} \\ -\mathscr{M}_{ik} & -0 & \mathscr{M}_{ii} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ji} & -\zeta_{ii} & 0 \\ \zeta_{jj} & -\zeta_{ij} & 0 \\ \zeta_{jk} & -\zeta_{ik} & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{ik} \\ \varepsilon_{jk} \\ \varepsilon_{kk} \end{bmatrix} \\
&+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} -\mathscr{M}_{jk} & 0 & \mathscr{M}_{ji} \\ -\mathscr{M}_{ik} & 0 & \mathscr{M}_{ii} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\zeta_{ji} & \zeta_{ii} & 0 \\ -\zeta_{jj} & \zeta_{ij} & 0 \\ -\zeta_{jk} & \zeta_{ik} & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{ii} \\ \varepsilon_{ji} \\ \varepsilon_{ki} \end{bmatrix} \\
&+(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} \mathscr{M}_{jj} & -\mathscr{M}_{ji} & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\zeta_{ki} & 0 & \zeta_{ii} \\ -\zeta_{kj} & 0 & \zeta_{ij} \\ -\zeta_{kk} & 0 & \zeta_{ik} \end{bmatrix}\begin{bmatrix} \varepsilon_{ik} \\ \varepsilon_{jk} \\ \varepsilon_{kk} \end{bmatrix} \\
&+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} \mathscr{M}_{jj} & -\mathscr{M}_{ji} & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ki} & 0 & -\zeta_{ii} \\ \zeta_{kj} & 0 & -\zeta_{ij} \\ \zeta_{kk} & 0 & -\zeta_{ik} \end{bmatrix}\begin{bmatrix} \varepsilon_{ii} \\ \varepsilon_{ji} \\ \varepsilon_{ki} \end{bmatrix} \qquad (114)
\end{aligned}
$$

$$
\begin{aligned}
g_{ij}\,(i\neq j) =\ &(\mathscr{M}_{1i}\mathscr{M}_{2i}\mathscr{M}_{3i})(\mathscr{K})(\varepsilon_{j1}\varepsilon_{j2}\varepsilon_{j3})^{T}+(\mathscr{M}_{1j}\mathscr{M}_{2j}\mathscr{M}_{3j})(\mathscr{K})(\varepsilon_{i1}\varepsilon_{i2}\varepsilon_{i3})^{T}+(\varepsilon_{1i}\varepsilon_{2i}\varepsilon_{3i})(Z)(\mathscr{M}_{j1}\mathscr{M}_{j2}\mathscr{M}_{j3})^{T} \\
&+(\varepsilon_{1j}\varepsilon_{2j}\varepsilon_{3j})(Z)(\mathscr{M}_{i1}\mathscr{M}_{i2}\mathscr{M}_{i3})^{T} \\
&+(\eta_{i}\,\eta_{ji}\eta_{ki})\begin{bmatrix} \mathscr{M}_{kj} & -\mathscr{M}_{ki} & 0 \\ 0 & 0 & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \end{bmatrix}\begin{bmatrix} 0 & \zeta_{ki} & -\zeta_{ji} \\ 0 & \zeta_{kj} & -\zeta_{jj} \\ 0 & \zeta_{kk} & -\zeta_{jk} \end{bmatrix}\begin{bmatrix} \varepsilon_{ij} \\ \varepsilon_{jj} \\ \varepsilon_{kj} \end{bmatrix} \\
&+(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix} \mathscr{M}_{kj} & -\mathscr{M}_{ki} & 0 \\ 0 & 0 & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\zeta_{ki} & \zeta_{ji} \\ 0 & -\zeta_{kj} & \zeta_{jj} \\ 0 & -\zeta_{jk} & \zeta_{jk} \end{bmatrix}\begin{bmatrix} \varepsilon_{i} \\ \varepsilon_{ji} \\ \varepsilon_{ki} \end{bmatrix} \\
&+(\eta_{ii}\eta_{ji}\eta_{ki})\left\{\begin{bmatrix} 0 & \mathscr{M}_{kk} & -\mathscr{M}_{kj} \\ 0 & 0 & 0 \\ 0 & -\mathscr{M}_{ik} & \mathscr{M}_{ij} \end{bmatrix}+\begin{bmatrix} 0 & 0 & 0 \\ M_{kk} & 0 & -M_{ki} \\ -M_{jk} & 0 & M_{ji} \end{bmatrix}\right\}\begin{bmatrix} -\zeta_{ji} & \zeta_{ii} & 0 \\ -\zeta_{jj} & \zeta_{ij} & 0 \\ -\zeta_{jk} & \zeta_{ik} & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{ij} \\ \varepsilon_{jj} \\ \varepsilon_{kj} \end{bmatrix} \\
&+(\eta_{ij}\eta_{jj}\eta_{kj})\left\{\begin{bmatrix} 0 & \mathscr{M}_{kk} & -\mathscr{M}_{kj} \\ 0 & 0 & 0 \\ 0 & -\mathscr{M}_{1k} & \mathscr{M}_{ij} \end{bmatrix}+\begin{bmatrix} 0 & 0 & 0\ 0 \\ M_{kk} & 0 & -M_{ki} \\ -0 & 0 & M_{ji} \end{bmatrix}\right\}\begin{bmatrix} \zeta_{ji} & -\zeta_{ii} & 0 \\ \zeta_{jj} & -\zeta_{ij} & 0 \\ \zeta_{jk} & -\zeta_{ik} & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{i} \\ \varepsilon_{ji} \\ \varepsilon_{ki} \end{bmatrix} \\
&+(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} 0 & 0 & 0 \\ \mathscr{M}_{kj} & -\mathscr{M}_{ki} & 0 \\ -\mathscr{M}_{jj} & \mathscr{M}_{ji} & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ki} & 0 & -\zeta_{ii} \\ \zeta_{kj} & 0 & -\zeta_{ij} \\ \zeta_{kk} & 0 & -\zeta_{ik} \end{bmatrix}\begin{bmatrix} \varepsilon_{ij} \\ \varepsilon_{jj} \\ \varepsilon_{kj} \end{bmatrix} \\
&+(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix} 0 & 0 & 0 \\ \mathscr{M}_{kj} & -\mathscr{M}_{ki} & 0 \\ -\mathscr{M}_{jj} & \mathscr{M}_{ji} & 0 \end{bmatrix}\begin{bmatrix} -\zeta_{ki} & 0 & \zeta_{ii} \\ -\zeta_{kj} & 0 & \zeta_{ij} \\ -\zeta_{kk} & 0 & \zeta_{ik} \end{bmatrix}\begin{bmatrix} \varepsilon_{ii} \\ \varepsilon_{ji} \\ \varepsilon_{ki} \end{bmatrix} \\
&+(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} \mathscr{M}_{jj} & -\mathscr{M}_{ji} & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\zeta_{ki} & \zeta_{ji} \\ 0 & -\zeta_{kj} & \zeta_{jj} \\ 0 & -\zeta_{kk} & \zeta_{jk} \end{bmatrix}\begin{bmatrix} \varepsilon_{ik} \\ \varepsilon_{jk} \\ \varepsilon_{kk} \end{bmatrix} \\
&+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} \mathscr{M}_{jj} & -\mathscr{M}_{ji} & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \zeta_{ki} & -\zeta_{ji} \\ 0 & \zeta_{kj} & -\zeta_{jj} \\ 0 & \zeta_{kk} & -\zeta_{jk} \end{bmatrix}\begin{bmatrix} \varepsilon_{ii} \\ \varepsilon_{ji} \\ \varepsilon_{ki} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+(\eta_{ii}\eta_{ji}\eta_{ki})\begin{bmatrix} 0 & \mathscr{M}_{jk} & -\mathscr{M}_{jj} \\ 0 & -\mathscr{M}_{ik} & \mathscr{M}_{ij} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ji} & -\zeta_{ii} & 0 \\ \zeta_{jj} & -\zeta_{ij} & 0 \\ \zeta_{jk} & -\zeta_{ik} & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{ik} \\ \varepsilon_{jk} \\ \varepsilon_{kk} \end{bmatrix} \\
&+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} 0 & \mathscr{M}_{jk} & -\mathscr{M}_{jj} \\ 0 & -\mathscr{M}_{ik} & \mathscr{M}_{ij} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\zeta_{ji} & \zeta_{ii} & 0 \\ -\zeta_{jj} & \zeta_{ij} & 0 \\ -\zeta_{jk} & \zeta_{ik} & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{ii} \\ \varepsilon_{ji} \\ \varepsilon_{ki} \end{bmatrix} \\
&+(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix} \mathscr{M}_{ij} & -\mathscr{M}_{ji} & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\zeta_{ki} & 0 & \zeta_{ii} \\ -\zeta_{kj} & 0 & \zeta_{ij} \\ -\zeta_{kk} & 0 & \zeta_{ik} \end{bmatrix}\begin{bmatrix} \varepsilon_{ik} \\ \varepsilon_{jk} \\ \varepsilon_{kk} \end{bmatrix} \\
&+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} \mathscr{M}_{jj} & -\mathscr{M}_{ji} & 0 \\ -\mathscr{M}_{ij} & \mathscr{M}_{ii} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ki} & 0 & -\zeta_{ii} \\ \zeta_{ij} & 0 & -\zeta_{ij} \\ \zeta_{ik} & 0 & -\zeta_{ik} \end{bmatrix}\begin{bmatrix} \varepsilon_{ij} \\ \varepsilon_{jj} \\ \varepsilon_{kj} \end{bmatrix} \\
&+(\eta_{ij}\eta_{jj}\eta_{kj})\begin{bmatrix} \mathscr{M}_{jk} & 0 & -\mathscr{M}_{ji} \\ -\mathscr{M}_{ik} & 0 & \mathscr{M}_{ii} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \zeta_{ji} & -\zeta_{ii} & 0 \\ \zeta_{jj} & -\zeta_{ij} & 0 \\ \zeta_{jk} & -\zeta_{ik} & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{ik} \\ \varepsilon_{jk} \\ \varepsilon_{kk} \end{bmatrix} \\
&+(\eta_{ik}\eta_{jk}\eta_{kk})\begin{bmatrix} \mathscr{M}_{jk} & 0 & -\mathscr{M}_{ji} \\ -\mathscr{M}_{ik} & 0 & \mathscr{M}_{ii} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\zeta_{ji} & \zeta_{ii} & 0 \\ -\zeta_{jj} & \zeta_{ij} & 0 \\ -\zeta_{jk} & \zeta_{ik} & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{ij} \\ \varepsilon_{jj} \\ \varepsilon_{kj} \end{bmatrix}
\end{aligned} \tag{115}
$$

$$
\begin{aligned}
\Pi = \underline{n}^T \Bigg\{ &-\begin{bmatrix} E_{22}+E_{33} & -E_{12} & -E_{13} \\ -E_{21} & E_{33}+E_{11} & -E_{23} \\ -E_{31} & -E_{32} & E_{11}+E_{22} \end{bmatrix} + \begin{bmatrix} M_{23}E_{23}+M_{32}E_{32} & M_{12}E_{33} & M_{13}E_{22} \\ M_{21}E_{33} & M_{31}E_{31}+M_{13}E_{13} & M_{23}E_{11} \\ M_{31}E_{22} & M_{32}E_{11} & M_{12}E_{12}+M_{21}E_{21} \end{bmatrix} \\
&+\begin{bmatrix} 0 & M_{13}E_{23}+M_{23}E_{13} + M_{31}E_{32}+M_{32}E_{31} & M_{12}E_{32}+M_{32}E_{12} + M_{21}E_{23}+M_{23}E_{21} \\ & 0 & M_{21}E_{31}+M_{31}E_{21} + M_{12}E_{13}+M_{13}E_{12} \\ & & 0 \end{bmatrix} \Bigg\} \underline{n}
\end{aligned} \tag{116}
$$

$$
\begin{aligned}
\Sigma = n_1[&(\mathscr{H}_{23}-\mathscr{H}_{32})\det(\underline{\underline{M}})\det(\underline{\underline{\zeta}}) + \{(\zeta_{32},\ -\zeta_{31},\ 0)+(-\zeta_{23},\ 0,\ \zeta_{21})\}(\mathscr{M})^T(\mathscr{H})(\eta)^T(M)(\zeta_{11},\ \zeta_{12},\ \zeta_{13})^T \\
&+n_2[(\mathscr{H}_{31}-\mathscr{H}_{13})\det(\underline{\underline{M}})\det(\underline{\underline{\zeta}}) + (0,\ -\zeta_{23},\ \zeta_{22})(\mathscr{M})^T(\mathscr{H})(\eta)^T(M)(\zeta_{11},\ \zeta_{12},\ \zeta_{13})^T \\
&+(Z_{11},\ Z_{12},\ Z_{13})(\mathscr{M})^T(\mathscr{H})(\eta)^T(M_{13},\ M_{23},\ M_{33})^T] + n_3[(\mathscr{H}_{12}-\mathscr{H}_{21})\det(\underline{\underline{M}})\det(\underline{\underline{\zeta}}) \\
&+(0,\ -\zeta_{33},\ \zeta_{32})(\mathscr{M})^T(\mathscr{H})(\eta)^T(M)(\zeta_{11},\ \zeta_{12},\ \zeta_{13})^T - (Z_{11},\ Z_{12},\ Z_{13})(\mathscr{M})^T(\mathscr{H})(\eta)^T(M_{12},\ M_{22},\ M_{32})^T]
\end{aligned} \tag{117}
$$

$$
\begin{aligned}
\Upsilon = n_1[&(\mathscr{M}_{11}\mathscr{M}_{12}\mathscr{M}_{13})(Z)^T(\varepsilon)\{(\eta_{32}\ -\eta_{31}\ 0)^T+(-\eta_{23}\ 0\ \eta_{21})^T\} + \{(\zeta_{23}\ -\zeta_{13}\ 0)+(-\zeta_{32}\ 0\ \zeta_{12})\}(\varepsilon)(\mathscr{H})^T(\mathscr{M}_{11}\mathscr{M}_{21}\mathscr{M}_{31})^T \\
&+(\mathscr{M}_{21}\mathscr{M}_{22}\mathscr{M}_{23})(Z)^T(\varepsilon)(\eta_{13}\ 0\ -\eta_{11})^T+(\zeta_{31}\ 0\ -\zeta_{11})(\varepsilon)(\mathscr{H})^T(\mathscr{M}_{12}\mathscr{M}_{22}\mathscr{M}_{32})^T \\
&+(\mathscr{M}_{31}\mathscr{M}_{32}\mathscr{M}_{33})(Z)^T(\varepsilon)(-\eta_{12}\ \eta_{11}\ 0)^T+(-\zeta_{21}\ \zeta_{11}\ 0)(\varepsilon)(\mathscr{H})^T(\mathscr{M}_{13}\mathscr{M}_{23}\mathscr{M}_{33})^T] \\
+n_2[&(\mathscr{M}_{21}\mathscr{M}_{22}\mathscr{M}_{23})(Z)^T(\varepsilon)\{(\eta_{32}\ -\eta_{31}\ 0)^T+(0\ \eta_{13}\ -\eta_{12})^T\} + \{(\zeta_{23}\ -\zeta_{13}\ 0)+(0\ \zeta_{31}\ -\zeta_{21})\}(\varepsilon)(\mathscr{H})^T(\mathscr{M}_{12}\mathscr{M}_{22}\mathscr{M}_{32})^T \\
&+(\mathscr{M}_{31}\mathscr{M}_{32}\mathscr{M}_{33})(Z)^T(\varepsilon)(-\eta_{22}\ \eta_{21}\ 0)^T+(-\zeta_{23}\ \zeta_{13}\ 0)(\varepsilon)(\mathscr{H})^T(\mathscr{M}_{13}\mathscr{M}_{23}\mathscr{M}_{33})^T \\
&+(\mathscr{M}_{11}\mathscr{M}_{12}\mathscr{M}_{13})(Z)^T(\varepsilon)(0\ -\eta_{23}\ \eta_{22})^T+(0\ -\zeta_{32}\ \zeta_{22})(\varepsilon)(\mathscr{H})^T(\mathscr{M}_{11}\mathscr{M}_{21}\mathscr{M}_{31})^T \\
+n_3[&(\mathscr{M}_{31}\mathscr{M}_{32}\mathscr{M}_{33})(Z)^T(\varepsilon)\{(-\eta_{23}\ 0\ \eta_{21})^T+(0\ \eta_{13}\ -\eta_{12})^T\} + \{(-\zeta_{32}\ 0\ \zeta_{12})+(0\ \zeta_{31}\ -\zeta_{21})\}(\varepsilon)(\mathscr{H})^T(\mathscr{M}_{13}\mathscr{M}_{23}\mathscr{M}_{33})^T \\
&+(\mathscr{M}_{11}\mathscr{M}_{12}\mathscr{M}_{13})(Z)^T(\varepsilon)(0\ -\eta_{33}\ \eta_{32})^T+(0\ -\zeta_{33}\ \zeta_{23})(\varepsilon)(\mathscr{H})(\mathscr{M}_{11}\mathscr{M}_{21}\mathscr{M}_{31})^T \\
&+(\mathscr{M}_{21}\mathscr{M}_{22}\mathscr{M}_{23})(Z)^T(\varepsilon)(\eta_{33}\ 0\ -\eta_{31})^T+(\zeta_{33}\ 0\ -\zeta_{13})(\varepsilon)(\mathscr{H})(\mathscr{M}_{12}\mathscr{M}_{22}\mathscr{M}_{32})^T]
\end{aligned} \tag{118}
$$

$$\Phi=\underline{n}^{T}\left\{\begin{bmatrix}0 & 0 & 0\\ -E_{31} & -E_{32} & -E_{33}\\ E_{21} & E_{22} & E_{23}\end{bmatrix}(\eta)^{T}\begin{bmatrix}M_{11}\\ M_{21}\\ M_{31}\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ M_{31} & M_{32} & M_{33}\\ -M_{21} & -M_{22} & -M_{23}\end{bmatrix}(\zeta)^{T}\begin{bmatrix}E_{11}\\ E_{21}\\ E_{31}\end{bmatrix}\right.$$

$$+\begin{bmatrix}E_{31} & E_{32} & E_{33}\\ 0 & 0 & 0\\ -E_{11} & -E_{12} & -E_{13}\end{bmatrix}(\eta)^{T}\begin{bmatrix}M_{12}\\ M_{22}\\ M_{32}\end{bmatrix}+\begin{bmatrix}-M_{31} & -M_{32} & -M_{33}\\ 0 & 0 & 0\\ M_{11} & M_{12} & M_{13}\end{bmatrix}(\zeta)^{T}\begin{bmatrix}E_{12}\\ E_{22}\\ E_{32}\end{bmatrix}$$

$$\left.+\begin{bmatrix}-E_{21} & -E_{22} & -E_{23}\\ E_{11} & E_{12} & E_{13}\\ 0 & 0 & 0\end{bmatrix}(\eta)^{T}\begin{bmatrix}M_{13}\\ M_{23}\\ M_{33}\end{bmatrix}+\begin{bmatrix}M_{21} & M_{22} & M_{23}\\ -M_{11} & -M_{12} & -M_{13}\\ 0 & 0 & 0\end{bmatrix}(\zeta)^{T}\begin{bmatrix}E_{13}\\ E_{23}\\ E_{33}\end{bmatrix}\right\} \tag{119}$$

$$X=-(\zeta_{11}\zeta_{12}\zeta_{13})(M)^{T}(\eta)(\mathscr{H})^{T}(\mathscr{M})(Z_{11}Z_{12}Z_{13})^{T} \tag{120}$$

$$\Psi=(\varepsilon_{11}\varepsilon_{12}\varepsilon_{13})(\mathscr{H})^{T}(\mathscr{M})(Z_{11}Z_{12}Z_{13})^{T}+(\varepsilon_{21}\varepsilon_{21}\varepsilon_{22}\varepsilon_{23})(\mathscr{H})^{T}(\mathscr{M})(Z_{21}Z_{22}Z_{23})^{T}+(\varepsilon_{31}\varepsilon_{32}\varepsilon_{33})(\mathscr{H})^{T}(\mathscr{M})(Z_{31}Z_{32}Z_{33})^{T} \tag{121}$$

$$\Omega=-(\eta_{11}\eta_{21}\eta_{31})(M)(\zeta)^{T}(E_{11}E_{21}E_{31})^{T}-(\eta_{12}\eta_{22}\eta_{32})(M)(\zeta)^{T}(E_{12}E_{22}E_{32})^{T}-(\eta_{13}\eta_{23}\eta_{33})(M)(\zeta)^{T}(E_{13}E_{23}E_{33})^{T} \tag{122}$$

3. 電氣型 및 磁氣型問題간의 analogy

Maxwell方程式 (70)과 (71), Helmholtz의 波動方程式 (74)와 (75), 波動演算子行列 (76)과 (77), 및 source函數 vector (78)과 (97)등을 比較하면 電氣型 및 磁氣型電磁場問題를 公式化하는데 있어서 서로간에 다음과 같은 密接한 類似性 즉 avalogy가 存在함을 알 수 있다.

$$\begin{aligned}
&\underline{\underline{W}}_{E}(\underline{k},\ \omega)\leftrightarrow-\underline{\underline{W}}_{M}(\underline{k},\ \omega), && \varepsilon_0\leftrightarrow\mu_0, && \underline{\underline{\varepsilon}}\leftrightarrow-\underline{\underline{\mu}}\\
&\underline{S}_{E}(\underline{k},\ \omega)\leftrightarrow-\underline{S}_{M}(\underline{k},\ \omega), && \omega\leftrightarrow\omega, && \underline{\underline{\eta}}\leftrightarrow-\underline{\underline{\zeta}}\\
&\underline{E}\ (\underline{k},\ \omega)\leftrightarrow\ \underline{H}\ (\underline{k},\ \omega), && k_0\leftrightarrow k_0, && \underline{\underline{E}}\leftrightarrow-\underline{\underline{M}}\\
&\underline{D}\ (\underline{k},\ \omega)\leftrightarrow-\underline{E}\ (\underline{k},\ \omega), && \underline{\underline{k}}\leftrightarrow\underline{\underline{k}}, && \underline{\underline{\mathscr{H}}}\leftrightarrow-\underline{\underline{Z}}\\
&\underline{J}\ (\underline{k},\ \omega)\leftrightarrow-\underline{M}\ (\underline{k},\ \omega), && && \underline{\underline{\mathscr{E}}}\leftrightarrow-\underline{\underline{\mathscr{M}}}
\end{aligned} \tag{123}$$

여기서 $\underline{\underline{E}}$와 $\underline{\underline{M}}$, $\underline{\underline{\mathscr{H}}}$와 $\underline{\underline{Z}}$ 및 $\underline{\underline{\mathscr{E}}}$와 $-\underline{\underline{\mathscr{M}}}$ 등은 式 (93)에서 定義된 行列이며 媒体 parameter의 逐次的除 因子次數가 漸次로 높아지는 순서로 配列되어 있다.

本 論文에서 誘導한 어떤 한型의 表現式중의 모든 量은 위 analogy對照表上의 相應한 量으로 置換하면 다른 型의 相應하는 表現式을 얻을 수 있다. 이 原理은 波動演算子行列과 그 逆行列에도 적용되므로 이제 電氣型波動行列과 그 逆行列이 각각 式 (82)~(90) 및 式 (91)~(122)와 같이 알려졌으므로 磁氣型波動行列과 그 逆行列의 表現式을 求하는 일은 간단한 analogy法의 例에 불과하다.

VI. 相對論的公式의 應用例

II 3節에서 相對運動이 없을 때는 S系에 대한 媒体 parameter의 相對論的公式이 S′系에 대한 公式과 一致하게 된다는 것을 證明하였다. 波動演算子行列와 그 逆行列에 대해서도 같은 事實을 立證할 수 있다.

즉 相對運動이 없을 때는 $\beta=0$이므로 式 (68)로부터

$$\underline{\underline{\varepsilon}}=\underline{\underline{\varepsilon}},\quad \underline{\underline{\mu}}=\underline{\underline{I}},\quad \underline{\underline{\zeta}}=\underline{\underline{\eta}}=0 \tag{124}$$

따라서 式 (76), (77)로부터 波動演算子行列은

$$\underline{\underline{W}}_{E}=k_0^2\underline{\underline{\varepsilon}}+\underline{\underline{k}}\ \underline{\underline{k}} \tag{125}$$

$$\underline{\underline{W}}_{M}=k_0^2\underline{\underline{\mu}}+\underline{\underline{k}}\ \underline{\underline{\varepsilon}}^{-1}\underline{\underline{k}} \tag{126}$$

한편 式 (78), (79)로부터 source函數 vector는

$$\underline{S}_{E}=j\omega\mu_0\underline{J}-j\underline{\underline{k}}\ \underline{M} \tag{127}$$

$$\underline{S}_{M}=j\omega\varepsilon_0\underline{M}+j\underline{\underline{k}}\ \underline{J} \tag{128}$$

이 式들은 靜止系에서의 잘 알려진 式들과 一致한다.[21]

式 (124) 로부터

$$\det(\underline{\underline{\mu}})=1 \tag{129}$$

따라서 式 (93) 으로부터

$$(M)=(M_{ij})=\underline{\underline{I}} \tag{130}$$

$$(\mathscr{M})=(\mathscr{M}_{ij})=\underline{\underline{I}} \tag{131}$$

$$(\mathscr{K})=(\mathscr{K}_{ij})=0 \tag{132}$$

$$(Z)=(Z_{ij})=0 \tag{133}$$

그러므로 式 (94) ~ (106) 로부터

$$P_{ij}=Q_{ij}=R_{ij}=U_{ij}=V_{ij}=T_{ij}=0 \tag{134}$$

$$S_{ii}=(n_i n_j n_k)\begin{bmatrix} -\varepsilon_{jj}-\varepsilon_{kk} & 0 & 0 \\ 0 & -\varepsilon_{jj} & -\varepsilon_{jk} \\ 0 & -\varepsilon_{ki} & -\varepsilon_{kk} \end{bmatrix}\begin{bmatrix} n_i \\ n_j \\ n_k \end{bmatrix}$$

$$=-n_i^2(\varepsilon_{jj}+\varepsilon_{kk})-n_j^2\varepsilon_{jj}-n_k^2\varepsilon_{kk}-n_j n_k(\varepsilon_{jk}+\varepsilon_{kj}) \tag{135}$$

$$S_{ij}(i\neq j)=(n_i n_j n_k)\begin{bmatrix} \varepsilon_{ji} & 0 & 0 \\ -\varepsilon_{kk} & \varepsilon_{ji} & 0 \\ \varepsilon_{jk} & \varepsilon_{ki} & 0 \end{bmatrix}\begin{bmatrix} n_i \\ n_j \\ n_k \end{bmatrix}$$

$$=n_i^2\varepsilon_{ji}+n_j^2\varepsilon_{ji}-n_j n_i\varepsilon_{kk}+n_k n_j\varepsilon_{ki}+n_k n_j\varepsilon_{jk} \tag{136}$$

式 (129) ~ (136) 을 式 (90) 에 代入하면

$$\mathrm{cof}(W_{Eij})=k^4 n_i n_j-k^2k_0^2S_{ij}+k_0^4E_{ij} \tag{137}$$

한편 波動行列의 行列式에 대해서는 式 (108) ~ (122) 로부터

$$\Gamma=\Theta=\Lambda=\Xi=0=\Sigma=\Upsilon=\Phi=X=\Psi=\Omega=0 \tag{138}$$

$$\Delta=\underline{n}^T\underline{\underline{\varepsilon}}\,\underline{n} \tag{139}$$

$$\Pi=-\underline{n}^T\begin{bmatrix} E_{22}+E_{33} & -E_{12} & -E_{13} \\ -E_{21} & E_{33}+E_{11} & -E_{23} \\ -E_{31} & -E_{32} & E_{11}+E_{22} \end{bmatrix}\underline{n} \tag{140}$$

式 (129) 과 式 (139) ~ (140) 을 式 (128) 에 代入하면

$$\det(\underline{\underline{W}}_E)=k^4k_0^2\underline{n}^T\underline{\underline{\varepsilon}}\,\underline{n}-k^2k_0^4\underline{n}^T\begin{bmatrix} E_{22}+E_{33} & -E_{12} & -E_{13} \\ -E_{21} & E_{33}+E_{11} & -E_{23} \\ -E_{31} & -E_{31} & E_{11}+E_{22} \end{bmatrix}\underline{n}+k_0^6\det(\underline{\underline{\varepsilon}}) \tag{139}$$

따라서 $\det(\underline{\underline{W}}_E)=0$라 놓면

$$k^4\underline{n}^T\underline{\underline{\varepsilon}}\,\underline{n}-k^2k_0^2\underline{n}^T\begin{bmatrix} E_{22}+E_{33} & -E_{12} & -E_{13} \\ -E_{21} & E_{33}+E_{11} & -E_{23} \\ -E_{31} & -E_{32} & E_{11}+E_{22} \end{bmatrix}\underline{n}+k_0^4\det(\underline{\underline{\varepsilon}})=0 \tag{140}$$

式 (140) 은 k^2에 관한 二次式으로서 잘 알려진 Appleton-Hertree式[22]의 가장 一般化된 表現式이다.

以上의 例로 表示된 바와 같이 本 論文에서 誘導한 相對論的 一般公式들은 適當한 特殊條件下에서 既知의 結果와 一致한다.

相對論的公式系와 非相對論的公式系間의 差異點은 到處에서 볼 수 있으나 顯著한 한例로서 Appleton-Hartree 方程式 $\det(\underline{\underline{W}}_E)=0$를 들 수 있다. 非相對論的 경우에는 前述한 바와 같이 式 (140)은 k의 四次式이나 k^2에 대해서는 二次式이므로 k^2에 대해서 2個의 根이 存在하며 2根은 각각 ordinary wave와 extraordinary wave를 代表한다.

한편 相對論的 경우에는 式 (128) 로부터 $\det(\underline{\underline{W}}_E)=0$란 方程式은 k의 四次式이기는 하나 k^2의 二次式은 아니다. 따라서 一般的으로 4個의 相異한 根이 存在하므로써 非相對論的경우의 波面에서 歪曲된 波面을 나타낸다. 그러

나 速度가 減少하여 零이되면 k의 四次式은 式(140)에 表示한 바와 같이 k^2의 二次式으로 縮退하고 波面의 歪曲은 消滅된다.

Ⅶ. 結 論

Magneto-plasma內를 定速度로 運動中인 source에 依한 電磁界問題를 定磁界와 相對運動의 方向이 任意方向에 놓여있는 一般的인 경우에 대해서 論하였다.

Minkowski의 關係式을 異方性分散媒体에서도 適用할 수 있게금 一般化했으며 誘導率等 媒体parameter의 相對論的 變換公式을 誘導하였다. 그 結果를 速度比 β의 多項式으로 整理하므로써 相對速度의 影響을 評價하는데 便利하도록 하였다.

電磁界가 만족하는 Helmholtz의 波動方程式도 異方性媒体에 대해서 一般化했으며 이를 行列形式으로 整理하였다. 電磁界는 波動演算子行列의 逆行列와 source函數 vector의 相乘數으로 表示되므로 波動行列의 逆行列을 求하여 그 結果를 具体的으로 表示하였다.

本 論文에서 誘導한 公式들은 一般的 式들로서 既知의 特殊한 條件下에서는 잘 알려진 結果과 一致함을 例示하였다.

波動演算子行列의 決定方程式은 電磁波의 屈折率과 傳播特性을 左右하는 바 相對運動時는 非相對運動때와는 달리 決定方程式은 k의 四次式이기는하나 k^2의 二次式은 아니기 때문에 波面에 歪曲을 招來한다.

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