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Energy Conversion in the Rossby Adjustment Process for Step-Like Initial Disturbances

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初期攪亂에 의한 로스비 變形過程에 있어서의 에너지 變換

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Abstract

Conversion of energy from potential to kinetic form is considered when a step-like initial disturbance is released to reach a final steady state. For small amplitute disturbances, linearization can be made and it is proved analytically that the conversion factor (ratio of generated kinetic to loss in potential energies) asymptotically approaches to 1/3 as the horizontal scale of disturbances becomes large.

要約：階段形의 初期海面攪亂이 定常狀態에 도달할 때에 생기는 에너지 變換(位置에너지로부터 運動에너지로)에 대하여 고찰하였다. 無限少 攪亂의 경우 線形化가 가능하며 이때 變換比(生成된 運動에너지의 消滅된 位置에너지에 대한)는 初期攪亂의 水平的 規模가 커짐에 따라 1/3에 近接함이 解析的으로 밝혀졌다.

INTRODUCTION

The Rossby adjustment process has provided a powerful insight in studying the effect of initial disturbances. The main physical constraints in this process are the conservation of potential vorticity and geostrophic balance. However, the non-linearity of the problem sometimes masks some physical processes occurring during the adjustment because only the initial and final states can be found.

Recent studies of the Rossby adjustment process (Gill, 1982, P. 194: Van Heijst, 1985: Seung, 1986) show that only 1/3 of the initial potential energy is converted into kinetic energy for initial disturbances of large horizontal extent. In the barotropic case, it can be shown that the rest (2/3) of energy is radiated away as propagating Pincare waves (Gill, 1982). In the baroclinic case (Van Heijst, 1985: Seung, 1986) where the initial disturbances are given as a density change of a layer, the problem cannot be linearized. Analytic methods lead to non-linear algebraic equations which are solved numerically for a given set of parameters. In those studies, the comparison of energies between the initial and final states leads to the same conversion factor (1/3). However, it is not known how the rest of the energy is lost. In all these cases, the energy of the final state was obtained only through the determination of velocities and elevation.

In this paper, we consider the energy conversion problem for simple cases; initial disturbances are given by step-like surface and interface elevations for barotropic and baroclinic cases respectively. Both potential and kinetic energies are obtained directly from the equation of motion. This can be done when the Fourier transform method is applied. This study then shows that the conversion factor can be found as a function of the horizontal scale of initial disturbances.

BAROTROPIC CASE

Assume an initial state where a part of the ocean surface is elevated uniformly by a small amount η_o. As shown in Fig. 1, this initial disturbance will later adjust itself to a geostrophic balance thus creating the geostrophic current. To simplify the problem, we consider a two dimensional case where conditions are uniform in the direction perpendicular to the page (same direction as the current, V). Denote that

H : undisturbed water depth
η : surface elevation in final state
L : extent of initial disturbance before the adjustment process.
b : final extent of the disturbed water columns after the adjustment process
ρ : density
f : Coriolis' parameter
g : gravity constant

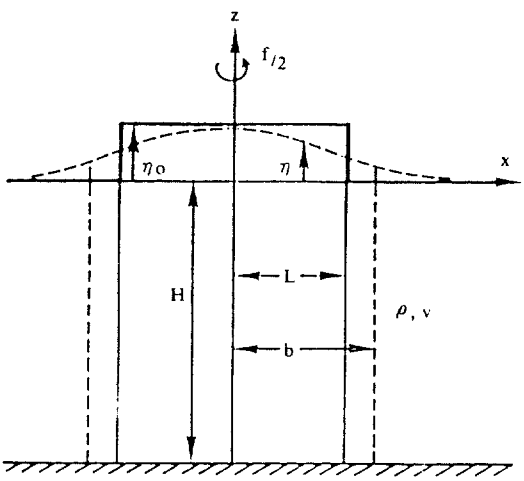

Fig. 1. Definition sketch of barotropic Rossby adjustment problem showing the initial (—) and final (--) states.

(x,z) : coordinate system

For conservation of potential vorticity of the disturbed ($|x|<b$) and undisturbed ($|x|>b$) water columns, the governing equations are

$$\frac{f}{H+\eta_0} = \frac{f+V_x}{H+\eta} \qquad \text{for } |x|<b \qquad (1)$$

$$\frac{f}{H} = \frac{f+V_x}{H+\eta} \qquad \text{for } |x|>b \qquad (2)$$

and for geostrophic balance, for all water columns,

$$fV = g\eta_x \qquad \text{for } -\infty<x<\infty \qquad (3)$$

In the above equations, subscript x denotes the differentiation with respect to x and b is yet to be found by solving the non-linear equations (1) through (3) in the same manner as Stommel and Veronis (1980), Van Heijst (1985) and Seung (1986).

We non-dimensionalize variables such that

$$(\eta, \eta_0) = H(\eta', \eta_0')$$
$$(L, b, x) = R_e(L', b', x')$$
$$V = fR_eV'$$

where $Re = \sqrt{gH}/f$ is the external Rossby radius. We drop the primes hereafter. Assuming small disturbances (η, $\eta_o \ll H$), linearization can be made to within O(η, η_o) because the unknown b can be approximated as L(*c.f.* Appendix). The final equation obtained from Eq.(1) through (3) is then given by

$$\eta_{xx} - \eta = -G(x) \qquad (4)$$

where

$$G(x) = \begin{cases} \eta_0 & \text{for } |x|<L \\ 0 & \text{for } |x|>L \end{cases} \qquad (5)$$

Since η_x is continuous, a Fourier transform can be made such that

$$\mathbf{F}(\eta) = \int_{-\infty}^{\infty} \eta e^{-ikx} dx$$

Applying this to Eq.(4), we obtain

$$F(\eta) = -\frac{F[G(x)]}{k^2+1} \qquad (6)$$

Equation (6) indicates that the response is dependent on the wave number. Since the spectral character of the final state (η) is dependent on the shape of the initial disturbance [G(x)] it can be said that the energy conversion cannot be generalized for arbitrary shapes of initial disturbance. It also shows that larger scale motion preserves more potential energy after the adjustment. For a step-like initial disturbance shown in Eq.(5), Eq(6) becomes

$$F(\eta) = -\frac{2\eta_0}{k^2+1}\frac{\sin kL}{k} \qquad (7)$$

and

$$F(V) = F(\eta_x) = ikF(\eta) \qquad (8)$$

Since

$$\int_{-\infty}^{\infty} \eta^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\eta)F^*(\eta)dk \qquad (9)$$

and

$$\int_{-\infty}^{\infty} V^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\eta_x)F^*(\eta_x)dk \qquad (10)$$

where * denotes complex conjugate, the change in (dimensional) potential energy

$$\Delta PE = \frac{\rho g H^2 R_e}{2}\left[\int_{-\infty}^{\infty} G^2(x)dx - \int_{-\infty}^{\infty}\eta^2 dx\right]$$

is given, with the use of Eqs. (5) and (7), by

$$\Delta PE = \frac{\rho g H^2 R_e \eta_0^2}{\pi}\int_{-\infty}^{\infty}\left[1 - \frac{1}{(1+k^2)^2}\right]\frac{\sin^2 kL}{k^2}dk \qquad (11)$$

The generated kinetic energy (dimensional)

$$KE = \frac{\rho H f^2 R_e^3}{2}\int_{-\infty}^{\infty} V^2 dx$$

is given, with the use of Eqs. (10), (8) and (7),

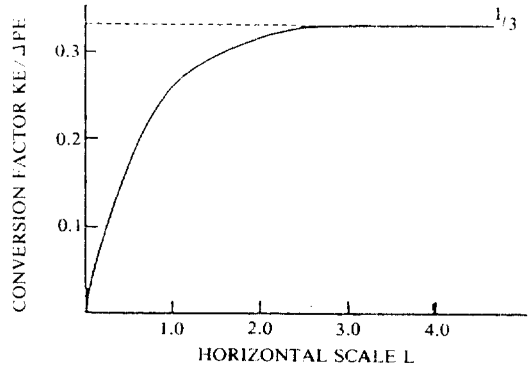

Fig. 2. The energy conversion factor as a function of the horizontal scale of initial disturbance. L is non-dimensionalized by the Rossby radius (external for barotropic and internal for baroclinic cases)

by

$$KE = \frac{\rho g H^2 R_e \eta_0^2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 kL}{(k^2+1)^2}dk \qquad (12)$$

to within O(η). Equations (11) and (12) give then the ratio (conversion factor)

$$\frac{KE}{\Delta PE} = \frac{1-e^{-2L}(1+2L)}{3-e^{-2L}(3+2L)} \qquad (13)$$

As shown in Fig. 2, this approaches asymptotically to 1/3 for large L.

BAROCLINIC CASE

The problem is basically same as the previous case. An initial disturbance is given on the density interface. Take a two layer ocean (Fig. 3) and denote again that

- H : undisturbed layer depth (same for both layers)
- η_0 : initial disturbance
- η : elevation in final state
- L : extent of initial disturbance before the adjustment process
- a(b) : final extent of the disturbed water columns in upper (lower) layer after the adjustment process
- $V_1(V_2)$: upper (lower) layer velocity
- $\rho_1(\rho_2)$: upper (lower) layer density

We non-dimensionalize variables such that

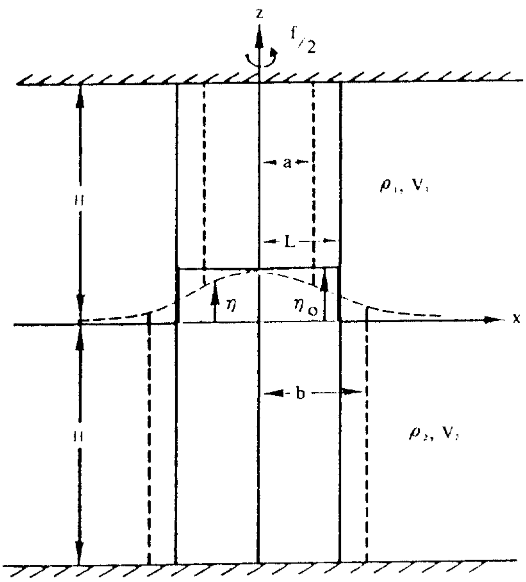

Fig. 3. Problem definition in baroclinic case. As in Fig. 1, solid and dotted lines represent the initial and final states respectively.

$$(\eta_0, \eta) = H(\eta_0', \eta')$$
$$(L, a, b, x) = R_i(L', a', b', x')$$
$$(V_1, V_2) = fR_i(V_1', V_2')$$

where $R_i = \sqrt{gH\Delta\rho/\rho_0}/|f|$ is the internal Rossby radius; $\Delta\rho = \rho_2 - \rho_1$; and ρ_0 is the mean density. Primes are dropped hereafter. For conservation of potential vorticity, the governing equations for each layer are

$$\frac{1}{1+\eta_0} = \frac{1+V_{2x}}{1+\eta}, \quad \frac{1}{1-\eta_0} = \frac{1+V_{1x}}{1-\eta} \qquad \text{for } |x|<a \tag{14}$$

$$\frac{1}{1+\eta_0} = \frac{1+V_{2x}}{1+\eta}, \quad 1 = \frac{1+V_{1x}}{1-\eta} \qquad \text{for } a<|x|<b \tag{15}$$

$$1 = \frac{1+V_{2x}}{1+\eta}, \quad 1 = \frac{1+V_{1x}}{1-\eta} \qquad \text{for } |x|>b \tag{16}$$

and for the thermal wind relationship,

$$V_1 - V_2 = -\eta_x \qquad \text{for } -\infty<x<\infty \tag{17}$$

By the same reasoning as before (c.f. Appendix), these equations can be linearized to within O(η, η_0) as follows:

$$\eta_{xx} - 2\eta = -2G(x) \tag{18}$$

where

$$G(x) = \begin{cases} \eta_0 & \text{for } |x|<L \\ 0 & \text{for } |x|>L \end{cases} \tag{19}$$

Since η_x is continuous, $F(\eta)$ can be given by

$$F(\eta) = \frac{-2F[G(x)]}{k^2+2} \tag{20}$$

The same arguement can be made about Eq.(20) as in the barotropic case, i.e. response is dependent on the shape and the horizontal scale of initial disturbance.

For an initial disturbance G(x) prescribed in Eq.(19), the changes in potential and kinetic energies are given by

$$\Delta PE = \frac{\Delta\rho g H^2 R_i}{2}\left[\int_{-\infty}^{\infty} G^2(x)dx - \int_{-\infty}^{\infty} \eta^2 dx\right] \tag{21}$$

and

$$KE \simeq \frac{\rho_0 H f^2 R_i^3}{2}\int_{-\infty}^{\infty}\frac{[(V_1-V_2)^2+(V_1+V_2)^2]}{2}dx = \frac{\Delta\rho g H^2 R_i}{4}\int_{-\infty}^{\infty}(\eta_x)^2 dx \tag{22}$$

respectively. Equation (22) is correct to within O(η_0, $\Delta\rho/\rho_0$). Note that the barotropic component, $(V_1+V_2)/2$, is absent. In the same way as before, these can be expressed in the wave number domain as follows:

$$\Delta PE = \frac{\Delta\rho g H^2 R_i \eta_0^2}{\pi}\int_{-\infty}^{\infty}\left[1-\frac{4}{(k^2+2)^2}\right]\frac{\sin^2 kL}{k^2}dk \tag{23}$$

$$KE = \frac{2\Delta\rho g H^2 R_i \eta_0^2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 kL}{(k^2+2)^2}dk \tag{24}$$

These lead to the ratio (conversion factor)

$$\frac{KE}{\Delta PE}=\frac{1-e^{-2\sqrt{2}L}(1+2\sqrt{2}L)}{3-e^{-2\sqrt{2}L}(3+2\sqrt{2}L)} \qquad (25)$$

which has the same character as that in the barotropic case (c.f. Fig. 2), i.e. asymptotically approaches to 1/3.

CONCLUSIONS

For step-like initial disturbances both in barotropic and baroclinic cases, the conversion factor approaches rapidly to 1/3 as their horizontal scale becomes large. This study also suggests that, for an initial disturbance given by surface (interface) elevation, the energy conversion cannot be generalized for arbitrary shapes of initial disturbance.

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APPENDIX

The mass conservation requires that

$$L(1+\eta_0)=\int_0^b (1+\eta)dx$$

Denoting $\bar{\eta}$ the mean value of η over the region $o<x<b$, this becomes

$$L(1+\eta_0)=(1+\bar{\eta})b$$

Reminding that $\bar{\eta}\,(\ll 1)$ is the same order of magnitude as $\eta_0\,(\ll 1)$, we can obtain the expression for b as

$$b=L[1+O(\eta_0)]$$

In the baroclinic case same arguments can be made for lower layer. Similarly, for upper layer, the mass conservation leads to

$$L(1-\eta_0)=\int_0^a (1-\eta)dx$$

For $\bar{\eta}$ as the mean value of η over $o<x<a$, we obtain

$$a=L[1+O(\eta_0)]$$

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