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Nonnull Distribution of the Determinant of B-Statistic in the Complex Case

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ABSTRACT

In this paper distribution of the determinant of B-statistic in the complex case has been derived in terms of incomplete gamma functions. Asymptotic expansion of the distribution has also been derived.

1. Introduction

Let the $p\times p$ random matrices A_1 and A_2 be independently distributed according to $CW_p(\cdot, m, \Sigma)$ and $CW_p(\cdot, n, \Sigma, \Theta)$ respectively *i.e.* A_1 is distributed as a central complex Wishart matrix and A_2 is distributed as noncentral complex Wishart with noncentrality parameter matrix Θ. Let the $p\times p$ random matrix $\tilde{L}$ be defined as

$$\tilde{L}=(A_1+A_2)^{-\frac{1}{2}}\ A_1\overline{(A_1+A_2)'}^{-\frac{1}{2}}.$$

The distribution of $\tilde{L}$ is known as noncentral complex multivariate beta type I.

The purpose of this article is to study the nonnull distributions of $|\tilde{L}|$ and $|I-\tilde{L}|$. First in section 2 some distributional results are derived. The exact and asymptotic distributions of a multiple of $-\ln|\tilde{L}|$ and $-\ln|I-\tilde{L}|$ have been derived in sections 3 and 4 respectively.

It may be noted that many authors have studied the distribution of random matrices, *e.g.* Roy (1966), Gupta (1971, 75, 76, 77), and Gupta et. al. (1975). The method used

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in this paper has been used previously by Nagarsenker (1979) and Gupta and Javier (1986).

2. Distribution of $\tilde{L}$

The hypergeometric function with complex matrix argument A is defined by

$$ {}_l\tilde{F}_t(a_1,\cdots,a_l;\ b_1,\cdots,b_t;A)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_l]_\kappa\tilde{C}_\kappa(A)}{[b_1]_\kappa\cdots[b_t]_\kappa\ \ k!} \tag{2.1}$$

where $\tilde{C}_\kappa$ called zonal polynomial (James (1964)) is a homogeneous polynomial of degree k in the latent roots of $A:\ p\times p$, which is a Hermitian matrix, $\kappa=(k_1,\cdots,k_p)$, $k_1\geq\cdots\geq k_p\geq 0$, $k_1+\cdots+k_p=k$; $\sum_\kappa$ is the summation over all such partitions; $a_1,\cdots,a_l$; $b_1,\cdots,b_t$ are real or complex constants with suitable restrictions on the b_j's;

$$[a]_\kappa=\tilde{\Gamma}_p(a,\kappa)/\tilde{\Gamma}_p(a)=\prod_{j=1}^{p}\frac{\Gamma[a+k_j-j+1]}{\Gamma(a-j+1)} \tag{2.2}$$

and

$$\begin{aligned}\tilde{\Gamma}_p(a)&=\int_{\bar{A}'=A>0}|A|^{a-p}\mathrm{etr}(-A)\,dA,\quad Re(a)>p-1\\ &=\pi^{p(p-1)/2}\prod_{j=1}^{p}\Gamma(a-j+1),\end{aligned} \tag{2.3}$$

is the complex multivariate gamma function. We now give the following two results needed in the sequel.

$$\int_{I>L=\bar{L}'>0}|L|^{a-p}|I-L|^{b-p}\tilde{C}_k(R(I-L))\,dL=\frac{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b,\kappa)}{\tilde{\Gamma}_p(a+b,\kappa)}\ \tilde{C}_\kappa(R),\quad Re(a,b)>p-1.$$

$$\int_{T=\bar{T}'>0}\mathrm{etr}(-ST)\ \ |T|^{b-p}\tilde{C}_k(RT)\,dT=\tilde{\Gamma}_p(b,\kappa)\,|S|^{-b}\tilde{C}_k(S^{-1}R).$$

The matrices R and S above are Hermitian positive definite. The following result regarding the density of $\tilde{L}$ is now easily derived.

Theorem 2.1: *If A_1 and A_2 are independently distributed according to $CW_p(\cdot,m,\Sigma)$ and $CW_p(\cdot,n,\Sigma,\Theta)$ respectively, then the p.d.f. of $\tilde{L}$ is given as*

$$\begin{aligned}f(\tilde{L})&=\{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(n)\}^{-1}|\Sigma|^{-(m+n)}\mathrm{etr}(-\Theta)\,|\tilde{L}|^{m-p}|I-\tilde{L}|^{n-p}\\ &\int_{T=\bar{T}'>0}\mathrm{etr}(-\Sigma^{-1}T)\,|T|^{m+n-p}{}_0\tilde{F}_1(n;\Theta\Sigma^{-1}T^{\frac{1}{2}}(I-\tilde{L})\ (\overline{T^{\frac{1}{2}}})'dT,\ 0<\tilde{L}<I.\end{aligned} \tag{2.4}$$

Proof : Let $\tilde{L}=T^{-\frac{1}{2}}A_1(\overline{T^{-\frac{1}{2}}})'$, $T=A_1+A_2$ in the joint distribution of A_1 and A_2. The jacobian of transformation is $J(A_1,A_2\rightarrow\tilde{L},T)=|T|^p$. Integration with respect to T gives the density of $\tilde{L}$. ∎

This distribution (2.4) is called the noncentral complex multivariate beta distribution of full rank.

Theorem 2.2: *If $\tilde{L}$ has noncentral complex multivariate beta distribution of full rank then*

$$E(|\tilde{L}|^h)=\frac{\tilde{\Gamma}_p(m+n)\tilde{\Gamma}_p(m+h)}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(m+n+h)}\cdot \operatorname{etr}(-\Theta)\,{}_1\tilde{F}_1(m+n,\ m+n+h;\ \Theta),$$
$$Re(h)>-m+p-1, \tag{2.5}$$

and

$$E(|I-\tilde{L}|^h)=\frac{\tilde{\Gamma}_p(m+n)\tilde{\Gamma}_p(n+h)}{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(m+n+h)}\operatorname{etr}(-\Theta)\,{}_2\tilde{F}_2(m+n,n+h;\ n,m+n+h;\Theta),$$
$$Re(h)>-n+p-1. \tag{2.6}$$

Proof: The results follow from the definition of expectation which gives integrals involving $\tilde{L}$ and T in both the cases. Integration of $\tilde{L}$ and T in that order yields the results (2.5) and (2.6). ▩

Lemma 2.1 : *For bounded h (Barnes (1899)),*

$$\ln \Gamma(x+h)=\ln\sqrt{2\pi}+(x+h-\tfrac{1}{2})\ln x-x+\sum_{r=1}^{m}\frac{(-1)^{r+1}B_{r+1}(h)}{r(r+1)x^r}+R_{m+1}(x) \tag{2.7}$$

where $R_m(x)$ is the remainder such that $|R_m(x)|\le a/|x|^m$, a is a constant independent of x, and $B_r(h)$ is the Bernoulli polynomial of degree r defined by $\gamma\, e^{h\gamma}(e^{\gamma}-1)^{-1}=\sum_{r=0}^{\infty}\frac{\gamma^r}{r!}B_r(h)$. These Bernoulli polynomials are extensively tabluated and for $r=1, 2, 3$ and 4 are given by (Anderson (1984)),

$$B_1(h)=h-\frac{1}{2},\quad B_2(h)=h^2-h+1/6,$$

$$B_3(h)=h^3-(3/2)h^2+\frac{1}{2}h,\quad B_4(h)=h^4-2h^3+h^2-1/30.$$

3. Distribution of $|\tilde{L}|$

Let $W=-2(m+n-u)\delta\ \ln|\tilde{L}|$, $0<W<\infty$, where u and δ (>0) are constants to be determined later, Then the characteristic function of W is given as

$$\phi_W(t)=E(e^{itW})=E[|\tilde{L}|^{-2(m+n-u)i\delta t}],$$

where $i=(-1)^{\frac{1}{2}}$. Using theorem 2.2 and (2.1), (2,2) and (2.3), we get

$$\phi_W(t)=\operatorname{etr}(-\Theta)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{\Gamma}_p(m+n,\kappa)}{\tilde{\Gamma}_p(m)}A_1(t,\kappa)\frac{\tilde{C}_\kappa(\Theta)}{k!}$$

where

$$A_1(t,\kappa) = \frac{\tilde{\Gamma}_p(m-2(m+n-u)i\delta t)}{\tilde{\Gamma}_p(m+n-2(m+n-u)i\delta t+k_j)}$$
$$= \prod_{j=1}^{p} \left\{ \frac{\Gamma[(m+n-u)(1-2i\delta t)+1-j-n+u]}{\Gamma[(m+n-u)(1-2i\delta t)+1-j+u+k_j]} \right\}. \quad (3.1)$$

Applying Barnes' formula (2.7) to (3.1), we get

$$\ln A_1(t,\kappa) = -(np+k)\ \ln[(m+n-u)(1-2i\delta t)] + \sum_{r=1}^{\infty} Q_r[(m+n-u)(1-2i\delta t)]^{-r},$$

where

$$Q_r = \frac{(-1)^{r+1}}{r(r+1)} \sum_{j=1}^{p} [B_{r+1}(1-j-n+u) - B_{r+1}(k_j+1-j+u)].$$

Hence we obtain

$$A_1(t,\kappa) = [(m+n-u)(1-2i\delta t)]^{-(np+k)}\ [1+\sum_{r=1}^{\infty} D_r^{\kappa}[(m+n-u)(1-2i\delta t)^{-r}], \quad (3.2)$$

where

$$D_r^{\kappa} = \frac{1}{r} \sum_{l=1}^{r} l\ Q_l D_{r-l}^{\kappa},\ \ D_0^{\kappa}=1,\ \ r=1,2,\cdots. \quad (3.3)$$

Thus letting $v=np+k$, we get

$$\phi_W(t) = \mathrm{etr}(-\Theta) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{\Gamma}_p(m+n,k)}{\tilde{\Gamma}_p(m)} \cdot \frac{\tilde{C}_\kappa(\Theta)}{k!}$$
$$[\{(m+n-u)^{-v}(1-2it\delta)^{-v}\} + \sum_{r=1}^{\infty} D_r^{\kappa}\{(m+n-u)(1-2i\delta t)\}^{-v-r}]. \quad (3.4)$$

Now by inverting the characteristic function (3.4), recognizing that $(1-2i\delta t)^{-a}$ is the characteristic function of the gamma density $g_a(2\delta,\ x)$, we obtain the *p.d.f.* of W in the following theorem.

Theorem 3.1 : *If $\tilde{L}$ is distributed according to a noncentral complex multivariate beta of full rank, then the p.d.f. of $W=-2\ (m+n-u)\delta\ \ln|\tilde{L}|$ is given by*

$$f_W(w) = \mathrm{etr}(-\Theta) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{\Gamma}_p(m+n,\kappa)}{\tilde{\Gamma}_p(m)} \frac{\tilde{C}_\kappa(\Theta)}{k!} \sum_{r=0}^{\infty} \frac{D_r^{\kappa}}{(m+n-u)^{r+v}}\ g_{r+v}(2\delta, w)$$

and the c.d.f. of W is given by

$$P[W \le w] = \mathrm{etr}(-\Theta) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{\Gamma}_p(m+n,\kappa)}{\tilde{\Gamma}_p(m)} \frac{\tilde{C}_\kappa(\Theta)}{k!} \sum_{r=0}^{\infty} \frac{D_r^{\kappa}}{(m+n-u)^{r+v}}\ G_{r+v}(2\delta, w) \quad (3.5)$$

where $G_a(2\delta,x) = \int_0^x g_a(2\delta,x)dx$ and the coefficients D_r^{κ} are defined by (3.3).

Next the distributon of $W^* = -2(m+n-u)\delta\ \ln|I-\tilde{L}|$ where u and δ (>0) are constants, is given in the following theorem.

Theorem 3.2 : *If $\tilde{L}$ is distributed as a noncentral complex multivariate beta of full rank, then the p.d.f. and the c.d.f. of $W^* = -2(m+n-u)\delta\ \ln|I-\tilde{L}|$ are given by*

$$f_{W^*}(w^*) = \text{etr}(-\Theta) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{\Gamma}_p(m+n, \kappa)}{\tilde{\Gamma}_p(n, \kappa)} \frac{\tilde{C}_\kappa(\Theta)}{k!} \sum_{r=0}^{\infty} J_r^\kappa (m+n-u)^{-(r+n)} g_{r+n}(2\delta, w^*)$$

and

$$P[W^* \leq w^*] = \text{etr}(-\Theta) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{\Gamma}_p(m, n, \kappa)}{\tilde{\Gamma}_p(n, \kappa)} \frac{\tilde{C}_\kappa(\Theta)}{k!} \sum_{r=0}^{\infty} J_r^\kappa (m+n-u)^{-(r+n)} G_{r+n}(2\delta, w^*) \tag{3.6}$$

where $n = mp$,

$$J_r^\kappa = \frac{1}{r} \sum_{l=1}^{r} l\, R_l\, J_{r-l}^\kappa, \quad J_0^\kappa = 1, \quad r = 1, 2, \cdots \tag{3.7}$$

and

$$R_l = \frac{(-1)^{l+1}}{l(l+1)} \sum_{j=1}^{p} [B_{l+1}(k_j+1-j-m+u) - B_{l+1}(k_j+1-j+u)]$$
$$l = 1, 2, \cdots. \tag{3.8}$$

Proof: Using (2.6), (2.1), (2.2) and (2.3), we get the characteristic function of W^* as

$$\phi_{W^*}(t) = \text{etr}(-\Theta) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{\Gamma}_p(m+n, \kappa)}{\tilde{\Gamma}_p(n, \kappa)} A_2(t, \kappa) \frac{\tilde{C}_\kappa(\Theta)}{k!} \tag{3.9}$$

where

$$A_2(t, \kappa) = \prod_{j=1}^{p} \left\{ \frac{\Gamma[(m+n-u)(1-2i\delta t) + k_j - j + 1 - m + u]}{\Gamma[(m+n-u)(1-2i\delta t) + k_j - j + 1 + u]} \right\}$$

Expanding logarithm of $A_2(t,k)$ by using Barnes' expansion, we get an expression in Bernoulli polynomials involving R_l given by (3.8). Inverting this expression back to $A_2(t,k)$ we get

$$A_2(t, \kappa) = [(m+n-u)(1-2i\delta t)]^{-n} [1 + \sum_{r=1}^{\infty} \{(m+n-u)(1-2i\delta t)\}^{-r} J_r^\kappa]$$

where J_r^κ is given by (3.7). Substituting the above expression in (3.9), one gets

$$\phi_{W^*}(t) = \text{etr}(-\Theta) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{\Gamma}_p(m+n, \kappa)}{\tilde{\Gamma}_p(n, \kappa) k!} \tilde{C}_\kappa(\theta) [\sum_{r=0}^{\infty} \{(m+n-u)(1-2i\delta t)\}^{-r-n} J_r^\kappa] \tag{3.10}$$

Inverting (3.10) yields the desired result. ▨

4. Asymptotic Distributions of $|\tilde{L}|$ and $|I-\tilde{L}|$

In this section we derive asymptotic expansion of the distribution of a suitable function of each of $|\tilde{L}|$ and $|I-\tilde{L}|$.

Theorem 4.1 : *The asymptotic distribution of* $W = -2(m+n-u)\delta \ln|\tilde{L}|$ *is given by*

$$P[W\leq w]=\sum_{k=0}^{\infty} P(k,\mathrm{tr}\Theta)G_{\nu}(2\delta,w)+\mathrm{etr}(-\Theta)\sum_{k=0}^{\infty}\sum_{\kappa}\omega_{1\kappa}[G_{1+\nu}(2\delta,w)$$

$$-G_{\nu}(2\delta,w)]\frac{\tilde{C}_{\kappa}(\Theta)}{k!}+O(m^{-2})$$

where $\nu=np+k$, $P(k,\mathrm{tr}\Theta)=\mathrm{etr}(-\Theta)[\mathrm{tr}\Theta]^{k}/k!$,

$$\omega_{1\kappa}=-\frac{1}{2\rho}\left[\frac{1}{m}\sum_{j=1}^{p}k_j(k_j-2j+1)+2k\left(1+\frac{n}{m}-\frac{\delta'}{\rho}\right)\right],\ \textit{and}\ \rho=\left[1+\frac{n-p}{2m}\right].$$

Proof : In the expression for the *c.d.f.* of W let

$$\psi(k)=\frac{\tilde{\Gamma}_p(m+n,k)}{\tilde{\Gamma}_p(m)}=\prod_{j=1}^{p}\left\{\frac{\Gamma[(m+n-u)+k_j+1+u-j]}{\Gamma[(m+n-u)+1+u-j-n]}\right\}$$

Using Barnes' formula (2.7), we have

$$\ln\psi(\kappa)=(np+k)\ \ln(m+n-u)+\sum_{r=1}^{\infty}Q'^{\kappa}_{r}(m+n-u)^{-r}$$

where for $r=1,2,\cdots$

$$Q'^{\kappa}_{r}=\frac{(-1)^{r+1}}{r(r+1)}\sum_{j=1}^{p}[B_{r+1}(k_j+u-j+1)-B_{r+1}\ (u-n-j+1)].$$

Thus

$$\psi(\kappa)=(m+n-u)^{np+k}[1+\sum_{r=1}^{\infty}D'^{\kappa}_{r}(m+n-u)^{-r}],\qquad(4.1)$$

where D'^{κ}_{r} is defined recursively by

$$D'^{\kappa}_{r}=\frac{1}{r}\sum_{l=1}^{r}lQ'^{\kappa}_{l}D'^{\kappa}_{r-l},\ \ D'^{\kappa}_{0}=1,\ \ r=1,2,\cdots.\qquad(4.2)$$

Substituting in (3.5) from (4.2), the *c.d.f.* of W can be written as

$$P[W\leq w]=\mathrm{etr}(-\Theta)\sum_{k=0}^{\infty}\sum_{\kappa}[\sum_{r=0}^{\infty}D'^{\kappa}_{r}\ (m+n-u)^{-r}]$$

$$\cdot[\sum_{r=0}^{\infty}D^{\kappa}_{r}(m+n-u)^{-r}G_{r+\nu}(2\delta,w)]\frac{\tilde{C}_{\kappa}(\Theta)}{k!}\qquad(4.3)$$

Multiplying out the two infinite series in square brackets in the last equation we get the product

$$G_{\nu}(2\delta,w)+[D'^{\kappa}_{1}G_{\nu}(2\delta,w)+D^{\kappa}_{1}G_{1+\nu}(2\delta,w)](m+n-u)^{-1}+O((m+n-u)^{-2}).$$

Since $D'^{\kappa}_{1}=-D^{\kappa}_{1}$ form equations (3.2) and (4.1), the expression (4.3) becomes

$$P[W\leq w]=\mathrm{etr}(-\Theta)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_{\kappa}(\Theta)}{k!}G_{\nu}(2\delta,w)$$

$$+\mathrm{etr}(-\Theta)\sum_{k=0}^{\infty}\sum_{\kappa}D^{\kappa}_{1}(m+n-u)^{-1}[G_{1+\nu}(2\delta,w)-G_{\nu}(2\delta,w)]$$

$$\cdot\frac{\tilde{C}_{\kappa}(\theta)}{k!}+O((m+n-u)^{-2})$$

$$=\sum_{k=0}^{\infty} P(k, tr\Theta) G_v(2\delta, w) + \text{etr}(-\Theta) \sum_{k=0}^{\infty} \sum_{\kappa} D_1^{\kappa} (m+n-u)^{-1} \frac{\tilde{C}_\kappa(\Theta)}{k!}$$

$$\cdot [G_{1+v}(2\delta, w) - G_v(2\delta, w)] + O((m+n-u)^{-2}) \qquad (4.4)$$

What remains to be done is the evaluation of D_1^{κ}, where

$$D_1^{\kappa} = \frac{1}{2} \sum_{j=1}^{p} [B_2(u-j-n+1) - B_2(k_j+u-j+1)].$$

After some algebra, we obtain

$$D_1^{\kappa} = -\frac{1}{2}\Big[\sum_{j=1}^{p} (k_j^2 - 2jk_j) + k(2u+1) + n(2pu - np - p^2)\Big] \qquad (4.5)$$

Now choose u so that $2pu - np - p^2 = 0$ and choose a number ρ' such that $(m+n-u)\delta = \rho' m$. Obtaining the solutions $u = (n+p)/2$ and $\rho' = \left(1 + \frac{n-p}{2m}\right)\delta$. Substituting these values in (4.5) we get

$$D_1^{\kappa} = -\frac{m}{2}\Big[\frac{1}{m}\sum_{j=1}^{p} (k_j^2 - 2jk_j + k_j) + 2k\left(1 + \frac{n}{m} - \frac{\rho'}{\delta}\right)\Big]$$

Substituting this last expression in (4.4) gives the desired result. ▨

Theorem 4.2 : *The asymptotic distribution of* $W^* = -2(m+n-u)\delta \ln|I - \tilde{L}|$ *is given by*

$$P[W^* \leq w^*] = G_\eta(2\delta, w^*) - \frac{m(\text{tr}\Theta + A)}{n\rho} [G_{\eta+1}(2\delta, w^*) - G_\eta] + O(n^{-2}) \quad (4.6)$$

where $\eta = mp$, $u = (m+p)/2 + (-A/p)$, A *is a real number and* $\rho = 1 + \frac{m-p}{2n} + \frac{A}{np}$.

Proof : In the expression (3.6) for the *c.d.f.* of W^*, let

$$\Delta(\kappa) = \frac{\tilde{\Gamma}_p(m+n, \kappa)}{\tilde{\Gamma}_p(n, \kappa)} = \prod_{j=1}^{p} \left\{\frac{\Gamma[(m+n-u) + k_j + 1 - j + u]}{\Gamma[(m+n-u) + k_j + 1 - j - m + u]}\right\}.$$

By Barnes' formula (2.7), the expansion in Bernoulli polynomials of this quantitiy is

$$\Delta(\kappa) = (m+n-u)^{\eta}[1 + \sum_{r=1}^{\infty} J'^{\kappa}_r (m+n-u)^{-r}], \qquad (4.7)$$

where $J'^{\kappa}_r = \frac{1}{r} \sum_{l=1}^{r} l R'_l J'^{\kappa}_{r}$, $J'^{\kappa}_0 = 1$, and

$$R'_l = \frac{(-1)^{l+1}}{l(l+1)} \sum_{j=1}^{p} [B_{l+1}(k_j + 1 - j + u) - B_{l+1}(k_j + 1 - j - m + u)] \quad l = 1, 2, \cdots.$$

Substituting (4.7) in (3.6) we get

$$P[W^* \leq w^*] = \text{etr}(-\Theta) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(\Theta)}{k!}\Big[\sum_{r=0}^{\infty} J'^{\kappa}_r (m+n-u)^{-r}\Big]$$

$$\cdot \Big[\sum_{r=0}^{\infty} J^{\kappa}_r (m+n-u)^{-r} G_{\eta+r}(2\delta, w^*)\Big] \qquad (4.8)$$

The product of the two infinite series enclosed in the square brackets is

$$G_\eta(2\delta, w^*) + J_1^{\kappa}[G_{1+\eta}(2\delta, w^*) - G_\eta(2\delta, w^*)](m+n-u)^{-1} + O((m+n-u)^{-2}) \qquad (4.9)$$

where $J_1^\kappa=\frac{1}{2}\sum_{j=1}^{p}[B_2(k_j+1-j-m+u)-B_2(k_j+1-j+u)]$. Simplifying this we get $J_1^\kappa=-\frac{m}{2}[2k-p^2+2pu-mp]$. Now choose u so that $2A-p^2+2pu-mp=0$ and $(m+n-u)\delta=\rho' n$. Obtaining the solutions $u=-A/p+(m+p)/2$ and $\rho'=\delta\left(1+\frac{m-p}{2n}+\frac{A}{np}\right)$. Substituting these values in (4.9) and subsequently putting these values in (4.8) we get

$$P[W^*\leq w^*]=\text{etr}(-\theta)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_\kappa(\theta)}{k!}[G_\eta(2\delta,w^*)-\frac{m(k-A)}{n\rho}\cdot\{G_{\eta+1}(2\delta,w^*)-G_\eta(2\delta,w^*)\}]+O(n^{-2}).$$

Notice that the factors in the square bracket in the above expression are independent of $\sum_\kappa$. Therefore using the result $\sum_\kappa \tilde{C}_k(\Theta)=\text{tr}\Theta$ and summing over k we can easily get (4.6). ▩

If we choose $A=\text{tr}\Theta$, then we get the following corollary of Theorem 4.2.

Corollary 4.2 : *The asymptotic distribution of* $W^*=-2(m+n-u)\delta\ln|I-\tilde{L}|$ *is given by*

$$P[W^*\leq w^*]=G_\eta(2\delta,w^*)+O(n^{-2})$$

when $u=(m+p)/2-tr\Theta/p$.

References

(1) Anderson, T.W. (1984). *An Introduction to Multivariate Statistical Analysis*, 2nd Edn. John Wiley, New York.

(2) Barnes, E.W. (1899). The theory of gamma functions. *Messeng. Math.*, 29, 64~129.

(3) Gupta, A.K. (1971). Noncentral distribution of Wilks' statistic in MANOVA. *Ann. Math. Statist.*, *42*, 1254~1261.

(4) Gupta, A.K, (1975). On a stochastic inequality for the Wilks' statistic. *Ann Inst. Statist. Math.*, *27*, 341~348.

(5) Gupta, A.K. (1976). Nonnull distribution of Wilks' statistic for MANOVA in the complex case. *Comn. Statist.*, *B5(4)*, 177~188.

(6) Gupta, A.K. (1977). On the sphericity test criterion in the multivariate Guassian distribution. *Aust. J. Statist.*, *19*, 202~205.

(7) Gupta, A.K., Chattopadhyay, A.K., and Krishnaiah, P.R. (1975). Asymptotic distributions of the determinants of some random matrices. *Comm. Statist.*, *4*(1), 33~47.

(8) Gupta, A.K. and Javier, W.R. (1986). Nonnull distribution of the determinant of B-statistic in multivariate analysis. *S. Afr. J. Statist.* (to appear)

(9) James. A.T. (1964). Distributions of matrix variates and latent roots derived from normal

samples. *Ann. Math. Statist.*, *35*, 475~500.

(10) Nagarsenker, B.N. (1979). Noncentral distribution of Wilks' statistic for tests of three hypotheses. *Sankhya*, *41*, Series A, Pts. 1 & 2, 67~81.

(11) Roy, J. (1966). Power of the likelihood-ratio test used in analysis of dispersion. *Multivariate Analysis*, Academic Press, New York.

(12) Singh, A. (1978). On the exact nonnull distribution of Wilks' L_{vc} criterion for the classical and complex normal populations and related problems. Department of Statistics, Purdue University, Mimeo Ser. No. 78—10.

⟨Corrections⟩

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Correction to "Nonnull Distribution of the Determinant of B-Statistic in the Complex Case", 15(1), 62~70 (1986)

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The following misprints have been left undetected during the editorial process. The editor sincerely apologizes for the errors. Corrected reprints of the paper are available with the authors and may be obtained on request.

Page	Line	Printed	Correction
63	17, 18	$\tilde{C}_k$	$\tilde{C}_\kappa$
	23	$\{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(n)\}^{-1}$	$\{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(n)\}^{-1}$
	24	$0<\tilde{L}<I$	$0<\tilde{L}=(\bar{\tilde{L}})'<I$
64	6	$\Gamma_p(m+n+h)$	$\tilde{\Gamma}_p(m+n+h)$
	24	,	.
	26	(2, 2)	(2. 2)
65	3	$\Gamma_p(m+n-2(m+n-u)i\delta t+k_j)$	$\Gamma_p(m+n-2(m+n-u)i\delta t, \kappa)$
	6	Q_γ	Q_r
	10	$D_r^\kappa[(m+n-u)(1-2i\delta t)^{-r}$	$D_r^\kappa[(m+n-u)(1-2i\delta t)]^{-r}$
	12	Ql	Q_l
	13	v	ν
	14	$\Gamma_p(m+n, k)$	$\Gamma_p(m+n, \kappa)$

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