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The Moore-Penrose Inverse for the Classificatory Models

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ABSTRACT

Many procedures for deriving the Moore-Penrose inverse X^+ have been developed, but the explicit forms of Moore-Penerose inverses for design matrices in analysis of variance models are not known heretofore. The purpose of this paper is to find explicit forms of X^+ for the one-way and the two-way analysis of variance models.

Consequently, the Moore-Penerose inverse X^+ and the shortest solutions of them can be easily obtained to the level of pocket calculator by way of our results.

1. Introduction

The application of generalized inverse matrices to linear statistical models is of relatively recent occurrence. In particular, the Moore-Penrose inverse of a matrix plays a significant role in statistics. In the statistical areas of analysis of variance and regression, this matrix characteristic is the basis for much of the mordern development. Many statisticians [e.g., Lowerre(1982), Kempthrone(1980), and Kennedy and Gentle(1980)] contributed to deriving the Moore-Penrose inverse X^+ for the special statistical models. But X^+ is not easy to compute, especially when X has many columns, and by this reason, the type of X^+ used in ANOVA is not known up to now.

The purpose of this paper is to provide general explicit forms X^+ for the classificatory models and shortest solutions. Also ANOVA tables are simply constructed without statistical computer package by use of the Moore-Penrose inverses.

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2. The Moore-Penrose Inverse

There is the special generalized inverse, X^+, of X called the Moore-Penrose inverse [Penrose(1955)]. It is the solution of the following equations:

(i) $XX^+X=X$,

(ii) $X^+XX^+=X^+$,

(iii) XX^+ is symmetric,

(iv) X^+X is symmetric. (2.1)

A procedure for deriving X^+ is to find $X'XU=X'$ and $XX'V=X$ and then $X^+=V'XU$. The following theorem will prove $V'XU$ become the Moore-Penrose inverse of X.

Theorem 2.1 If $X^+=V'XU$ such that $X'XU=X'$ and $XX'V=X$, then X^+ is the Moore-Penrose inverse of X that is X^+ satisfies the four conditions of (2.1).

Proof. From $X'XU=X'$ and $XUX=X$, XU is unique and symmetric, and also from $XX'V=X$, $X'VX'=X'$, $X'V$ is unique and symmetric. Therefore,

(i) $X(V'XU)X=XUX=X$.

(ii) $(V'XU)X(V'XU)=V'XV'XU=V'XU$.

(iii) $X(V'XU)=XU$, symmetric,

(iv) $(V'XU)X=V'X$, symmetric.

Thus $X^+=V'XU$ is the Moore-Penrose inverse of X. To see $V'XU$ is unique, let U_1 and U_2 be solutions to $X'XU_i=X'$, $i=1,2$ and V_1 and V_2 be solutions to $XX'V_i=X$, $i=1,2$. Since $XU_1=XU_2$ and $X'V_1=X'V_2$, so $V_1'XU_1=V_2'XU_1=V_2'XU_2$. ▩

The Moore-Penrose inverse is useful in experimental design and the analysis of covariance because the normal equations will not have a unique solution.

Definition 2.1 Given a consistent but overdetermined system of equation $y=X\beta$, a solution of this system b is called a *minimum-length solution* or *shortest solution* if b has the smallest euclidean norm among its solutions.

The shortest solution of the normal equations can be easily obtained and this solution is $b=X^+y$. To show this result, the following theorems and corollaries will be useful.

Theorem 2.2 $(X'X)^+=X^+(X^+)'$.

Proof. To show $X^+(X^+)'$ is the Moore-Penrose inverse, $X^+(X^+)'$ should satisfy the four conditions of (2.1).

(i) $X'X(X^+(X^+)')X'X=X'XX^+X^{+\prime}X'X=X'X^{+\prime}X'X^{+\prime}X'X=X'X^{+\prime}X'X=X'X.$

(ii) $X^+(X^+)'X'XX^+(X^+)'=X^+XX^+XX^+(X^+)'=X^+(X^+)'.$

(iii) $X'XX^+(X^+)'=X'(X^+)'X'(X^+)'=X'(X^+)'=X^+X=(X^+X)'.$

(iv) $X^+(X^+)'X'X=X^+XX^+X=X^+X=(X^+X)'.$

Hence $X^+(X^+)'$ is the Moore-Penrose inverse of $X'X$. ▨

Corollary 1 $(X^+)'=(X')^+$

Proof. We have $X^+=V'XU$, so $(X^+)'=U'X'V$. If we transpose the relations (i), (ii), (iii) and (iv) given in (2.1), we can obtain

(i) $X'(X^+)'X'=X'$,

(ii) $(X^+)'X'(X^+)'=(X^+)'$,

(iii) $XX^+=(X^+)'X'$,

(iv) $X^+X=X'(X^+)$

Hence $(X^+)'=(X')^+$. ▨

Theorem 2.3 If $Ax=b$ is consistent, then the shortest solution is $x=A^+b$.

Proof. Suppose x satisfies $Ax=b$. Consider A^+b, then of course $AA^+b=b$. Write

$$x=A^+b+(x-A^+b)=u+v, \text{ where } u=A^+b,\ v=x-A^+b.$$

Then

$$x'x=u'u+2u'v+v'v.$$

But

$$\begin{aligned} u'v&=b'A^{+\prime}(x-A^+b)\\ &=x'A'A^{+\prime}(x-A^+b)\\ &=x'A^+A(x-A^+b)\\ &=x'A^+(Ax-AA^+b)\\ &=x'A^+(b-b)=0. \end{aligned}$$

Thus $u'u\leqq x'x$. ▨

Corollary 2 Consider the normal equation $X'Xb=X'y$, then shortest solution is $b=X^+y$.

Proof. We already knew that $(X')^+=(X^+)'$ and $(X'X)^+=X^+(X^+)'$.

Also the normal equations are consistent. By the result of Theorem 2.3, the shortest solution is

$$b=(X'X)^+X'y=X^+(X')^+X'Y=X^+(X^+)'X'y=X^+XX^+y=X^+y.$$ ▨

3. The Moore-Penrose Inverse for the Classificatory Models

A. The one-way unequal classification.

The equation of the model for the one-way classification is

$$y_{ij}=\mu+\alpha_i+e_{ij},$$

where $i=1,2,\cdots,a$ and $j=1,2,\cdots,n_i$.

The general matrix formulation of the model for the one-way classification is

$$y=Xb+e,$$

where $b'=(\mu,\alpha_1,\cdots,\alpha_a)$, $y'=(y_{11},y_{12},\cdots,y_{an_a})$, $X=(1_N,\ \sum_{i=1}^{a}{}^{+}1_{n_i})$, Σ^+ represents a direct sum of matrices (e.g., Searle 1982, Sec. 10.6) and $N=\sum_{i=1}^{a}n_i$.

Result 3.1.1 The Moore-Penrose inverse of X for the one-way classification is

$$X^+_{(a+1)\times N}=\frac{1}{a+1}\begin{bmatrix} \frac{1}{n_1}1'_{n_1} & \frac{1}{n_2}1'_{n_2} & \cdots & \frac{1}{n_a}1'_{n_a} \\ \frac{a}{n_1}1'_{n_1} & -\frac{1}{n_2}1'_{n_2} & \cdots & -\frac{1}{n_a}1'_{n_a} \\ -\frac{1}{n_1}1'_{n_1} & \frac{a}{n_2}1'_{n_2} & \cdots & -\frac{1}{n_a}1'_{n_a} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{1}{n_1}1'_{n_1} & -\frac{1}{n_2}1'_{n_2} & \cdots & \frac{a}{n_a}1'_{n_a} \end{bmatrix}.$$

The detailed explanation for this derivation is given in Appendix A.1.

Result 3.1.2 The shortest solution of the normal equations for the unequal one-way classification is

$$b^*=X^+y=\begin{bmatrix} \sum_{i=1}^{a}\bar{y}_{i.}/(a+1) \\ \\ \bar{y}_{i.}-\sum_{i=1}^{a}\bar{y}_{i.}/(a+1) \end{bmatrix},$$

where $i=1,2,\cdots,a$, and $\bar{y}_{i.}$ represent the average of the observations under the ith treatment.

B. The two-way classification with no interaction and k observations per cell.

The equation of the model for the two-way classification with no interaction and k

observations per cell is

$$y_{ijp}=\mu+\alpha_i+\beta_j+e_{ijp}\ ,$$

where $i=1, 2, \cdots, r,\ \ j=1, 2, \cdots, c$ and $p=1, 2, \cdots, k$.

The general matrix formulation of the model for the two-way classification with no interaction and k observations per cell is

$$y=1_N u+X_r r+X_c c+e,$$

where

$$r'=(\alpha_1, \alpha_2, \cdots, \alpha_r),$$

$$c'=(\beta_1, \cdots, \beta_c),$$

$$X_r=\sum_{i=1}^{r}{}^{+}1_{ck},$$

$$X_c'=\left(\sum_{i=1}^{c}{}^{+}1_k',\ \sum_{i=1}^{c}{}^{+}1_k', \cdots,\ \sum_{i=1}^{c}{}^{+}1_k'\right)_{c\times rck}\ ,$$

and $N=rck$.

First, the case of one observation per cell in two way classification with no interaction will be discussed.

Result 3.2.1 The Moore-Penrose inverse of X for the two-way classification with no interaction and one observation per cell is

$$X^{+}_{(r+c+1)\times(rc)}=\frac{1}{cr+c+r}\begin{bmatrix} 1_c' & 1_c' & 1_c' & \cdots & 1_c' \\ \frac{cr+r-1}{c}1_c' & -\frac{(c+1)}{c}1_c' & -\frac{(c+1)}{c}1_c' & \cdots & -\frac{(c+1)}{c}1_c' \\ -\frac{(c+1)}{c}1_c' & \frac{cr+r-1}{c}1_c' & -\frac{(c+1)}{c}1_c' & \cdots & -\frac{(c+1)}{c}1_c' \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -\frac{(c+1)}{c}1_c' & -\frac{(c+1)}{c}1_c' & -\frac{(c+1)}{c}1_c' & \cdots & \frac{cr+r-1}{c}1_c' \\ K & K & K & \cdots & K \end{bmatrix}$$

where $K=(c+2)\ I_c\ -\frac{r+1}{r}\ J_c$ and J_c is $c\times c$ matrix. The detailed derivation for this result is given in Appendix A.2.

Result 3.2.2 The shortest solution for the two-way classification with no interaction and one observation per cell is

$$b^*=X^+y=\begin{bmatrix} Y_{..}/(rc+r+c) & \\ Y_{i.}/c-\frac{c+1}{c}Y_{..}/(rc+r+c) & i=1, 2, \cdots, r \\ Y_{.j}/r-\frac{r+1}{r}Y_{..}/(rc+r+c) & j=1, 2, \cdots, c \end{bmatrix}$$

Next the case of two-way classification with no interaction and without interaction and k observations per cell will be represented as follows.

Result 3.3.1

The Moore-Penrose inverse of X for the two way classification without interaction and k observations per cell is

$$X^{+}=\begin{bmatrix} 1_5' & 1_5' & \cdots & 1_5' \\ \frac{cr+r-1}{c}1_5' & -\frac{c+1}{c}1_5' & \cdots & -\frac{c+1}{c}1_5' \\ -\frac{c+1}{c}1_5' & \frac{cr+r-1}{c}1_5' & \cdots & -\frac{c+1}{c}1_5' \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{c+1}{c}1_5' & -\frac{c+1}{c}1_5' & \cdots & \frac{cr+r-1}{c}1_5' \\ H_1 & H_1 & \cdots & H_1 \end{bmatrix},$$

$$H_1=\begin{bmatrix} \frac{cr+r-1}{r}1_3' & -\frac{r-1}{r}1_3' & \cdots & -\frac{r-1}{r}1_3' \\ -\frac{r-1}{r}1_3' & \frac{cr+r-1}{r}1_3' & \cdots & -\frac{r-1}{r}1_3' \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{r-1}{r}1_3' & -\frac{r-1}{r}1_3' & \cdots & \frac{rc+r-1}{r}1_3' \end{bmatrix},$$

where 1_5 is $ck\times 1$ matrix and 1_3 is $c\times 1$ matrix. The detailed derivation for this result is given in Appendix A.3.

Result 3.3.2 The shortest solution for the two-way classification with no interaction and k observations per cell is

$$b^{*}=X^{+}y=\begin{bmatrix} Y_{\cdots}/(k(rc+r+c)) & \\ Y_{i\cdot\cdot}/(ck)-\frac{c+1}{c}\frac{Y_{\cdots}}{k(rc+r+c)} & i=1,2,\cdots,r \\ Y_{\cdot j\cdot}/(rk)-\frac{r+1}{r}\frac{Y_{\cdots}}{k(rc+r+c)} & j=1,2,\cdots,c \end{bmatrix}.$$

C. The two-way classification with interaction and unequal number of observations in the cells without missing observations.

The general consideration for this model is

$$y_{ijp}=u+\alpha_i+\beta_j+\alpha\beta_{ij}+e_{ijp},$$

where $i=1,\cdots,r,\ j=1,\cdots,c,\ p=1,\cdots,n_{ij}$, where $n_{ij}>0$.

The general matrix formulation for the two way classification with interaction and unequal numbers of observations in the cells without missing observations is

$$y = 1_N u + X_r r + X_c c + X_{rc} rc + e,$$

where $r' = (\alpha_1, \cdots, \alpha_r)$, $c' = (\beta_1, \cdots, \beta_c)$,

$$rc' = (\alpha\beta_{11}, \alpha\beta_{12}, \cdots, \alpha\beta_{rc}),$$

$$X_r = \sum_{i=1}^{r}{}^{+} 1_{n_{i.}},$$

$$X_c' = \left(\sum_{j=1}^{c}{}^{+} 1'_{n_{1j}}, \sum_{j=1}^{c}{}^{+} 1'_{n_{2j}}, \cdots, \sum_{j=1}^{c}{}^{+} 1'_{n_{rj}}\right),$$

$$X_{rc}' = \sum_{p=1}^{rc}{}^{+}\left(\sum_{i=1}^{r}\sum_{j=1}^{c} 1'_{n_{ij}}\right), \text{ and } N = \sum_{i=1}^{r}\sum_{j=1}^{c} n_{ij}.$$

Before discussing about the unequal numbers of observations case, first we will consider the case of one observation per cell.

Result 3.4.1 The Moore-Penrose inverse of X for the two-way classification with interaction and one observation per cell is

$$X^{+} = \frac{1}{(r+1)(c+1)} \left[\begin{array}{c}
1'_{rc} \\ \hline
r1_c' \;\Big|\; -1'_{c(r-1)} \\ \hline
-1_c' \;\Big|\; r1_c' \;\Big|\; -1_c'{}_{(r-2)} \\ \hline
\vdots \\ \hline
-1_c'{}_{(r-1)} \;\Big|\; r1_c' \\ \hline
\sum_{j=1}^{r}{}_{+}\{(c+1)I_c - J_c\} \\ \hline
r\{(c+1)I_c - J_c\} \;\Big|\; -\sum_{i=1}^{r-1}{}_{+}\{(c+1)I_c - J_c\} \\ \hline
-(c+1)I_c + J_c \;\Big|\; r\{(c+1)I_c - J_c\} \;\Big|\; -\sum_{i=1}^{r-2}{}_{+}\{(c+1)I_c - J_c\} \\ \hline
\vdots \\ \hline
-\sum_{i=1}^{r-1}{}_{+}\{(c+1)I_c - J_c\} \;\Big|\; r\{(c+1)I_c - J_c\}
\end{array}\right],$$

where $\sum_{i=1}^{r}{}_{+}(aI_c + bJ_c) = (aI_c + bJ_c, \cdots, aI_c + bJ_c)_{c \times cr}$. The detailed derivation for this result is given in Appendix A.4.

Using this result, the shortest solutions of the two-way classification with interaction and one observation per cell can be easily obtained as follows.

Result 3.4.2 The shortest solution of the normal equations for two-way classification with interaction and one observation per cell is

$$b^*=X^+y=\begin{bmatrix} \frac{1}{(r+1)(c+1)}y_{..} \\ \frac{1}{c+1}y_{i.}-\frac{1}{(r+1)(c+1)}y_{..} \\ \frac{1}{r+1}y_{.j}-\frac{1}{(r+1)(c+1)}y_{..} \\ y_{ij}-\frac{1}{r+1}y_{.j}-\frac{1}{c+1}y_{i.}+\frac{1}{(r+1)(c+1)}y_{..} \end{bmatrix}.$$

Now using previous case, the case of k observations per cell can be extended as follows.

Result 3.5.1 The Moore-Penrose inverse of X for the two-way classification with interaction and k observations per cell can be obtained through *the following algorithm.*

Step 1. Choose X^+ be Moore-Penrose inverse for the two-way classification with interaction and one observation per cell.

Step 2. For $i:=1, rc+r+c+1$ do

For $j:=1, rc-1$ do

(1) Divide each element a_{ij} of X by k, say it a_{ij}^*.

(2) Insert a_{ij}^* $(k-1)$ times between a_{ij}^* and $a_{i,j+1}$.

For $j=rc$ divide a_{ij} by k.

(3) Insert a_{ij}^* $(k-1)$ times at the right side of a_{ij}.

(* terminate i-th row operation and try $(i+1)$-th row operation until $i=rc+r+c+1$*)

Result 3.5.2 The shortest solution of the normal equations for the two way classification with interaction and k observations per cell is

$$b^*=X^+y=\begin{bmatrix} \frac{1}{k(r+1)(c+1)}y_{...} \\ \frac{1}{k(c+1)}y_{i..}-\frac{1}{k(r+1)(c+1)}y_{...} \\ \frac{1}{k(r+1)}y_{.j.}-\frac{1}{k(r+1)(c+1)}y_{...} \\ \frac{1}{k}y_{ij.}-\frac{1}{k(c+1)}y_{i..}-\frac{1}{k(r+1)}y_{.j.}+\frac{1}{k(r+1)(c+1)}y_{...} \\ \text{for } i=1,\cdots,r \ j=1,\cdots,c \end{bmatrix}.$$

Finally, we can find the results of unequal observations per cell.

Result 3.6.1 : The process of finding the Moore-Penrose inverse of X for the two way classification with interaction and n_{ij} observations per cell is the same as that of k observations per cell except using n_{ij} instead of k in Result 3.5.1.

Result 3.5.1 and result 3.6.1 can be obtained through similar procedure of Appendix A.4.

Result 3.6.2 The shortest solution of the normal equations for the two-way classification with interaction and n_{ij} observations per cell is

$$b^*=X^+y=\begin{bmatrix} \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{1}{(r+1)(c+1)n_{ij}}y_{ij\cdot} \quad : u^* \\ \sum_{j=1}^{c}\frac{r}{(r+1)(c+1)n_{pj}}y_{pj\cdot} \quad -\sum_{d\neq p}^{r}\sum_{j=1}^{c}\frac{1}{(r+1)(c+1)n_{dj}}y_{dj\cdot} : \alpha_p^* \\ \sum_{i=1}^{r}\frac{c}{n_{iq}(r+1)(c+1)}y_{iq\cdot}-\sum_{i=1}^{r}\sum_{j\neq q}^{c}\frac{1}{(r+1)(c+1)n_{ij}}y_{ij\cdot} : \beta_q^* \\ \frac{rc}{(r+1)(c+1)n_{pq}}y_{pq}-\sum_{j\neq q}^{c}\frac{r}{(r+1)(c+1)n_{pj}}y_{pj\cdot} \\ -\sum_{i\neq p}^{r}\frac{c}{(r+1)(c+1)n_{iq}}y_{iq\cdot} \\ +\sum_{i\neq p}^{r}\sum_{j\neq q}^{c}\frac{1}{(r+1)(c+1)n_{ij}}y_{ij\cdot} : (\alpha\beta)^*_{pq} \end{bmatrix}$$

Remark: The shortest solution can, of course, be obtained directly working the constrained least square problem: minimize $b'b$ subject to $X'Xb=X'y$.

5. Conclusions

Several results completely characterize the Moore-Penrose inverse of Matrix for one-way, two-way, and two-way classification with interaction models. Also the shortest solutions of their models are obtained. Our procedure is applicable to the nested design, and Latin square design. In the case of missing cells, it is not easy to obtained explicit form of the Moore-Penrose inverse.

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APPENDIX A

1. Derivation of the Moore-Penrose inverse of X for the one-way classification.

To get X^+, first find V such that $XX'V=X$.

$$(XX')_{N\times N}=\begin{bmatrix} 2J_{11} & J_{12} & \cdots & J_{1a} \\ J_{21} & 2J_{22} & \cdots & J_{2a} \\ \cdot & \cdot & \cdots & \cdot \\ J_{a1} & J_{a2} & \cdots & 2J_{aa} \end{bmatrix},$$

where J_{ij} is the $n_i \times n_j$ matrix whose every element is 1. To find a non-singular principal minor of order a (=rank X), select first row and column element from each J_{ij}.

Then this matrx is $M_{a\times a}=I+J$. Thus, M^{-1} becomes

$$M^{-1}=I-\frac{1}{a+1}J$$

$$=\frac{1}{a+1}\begin{bmatrix} a & -1 & -1 & \cdots & -1 \\ -1 & a & -1 & \cdots & -1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -1 & -1 & -1 & \cdots & a \end{bmatrix}.$$

Let X^* be the matrix selected the first row from each $n_i (i=1, 2, \cdots, a)$ in the matrix X. Then X^* becomes as follows

$$X^*_{a\times(a+1)}=[1_a, I_{a\times a}].$$

Thus

$$M^{-1}X^*=[I-(1/(a+1))J]\ [1_a, I_{a\times a}]$$
$$=(1/(a+1))[1_a, (a+1)I-J]$$
$$=V^*(\text{say}),$$

then

$$(V'^*X^*)_{(a+1)\times(a+1)}=\frac{1}{a+1}\begin{bmatrix} 1_a' \\ (a+1)I-J \end{bmatrix}[1_a, I_{a\times a}]$$
$$=\frac{1}{a+1}\begin{bmatrix} a & 1'_a \\ 1_a & (a+1)I-J \end{bmatrix}$$
$$=V'X.$$

Next, find U such that $X'XU=X'$.

$$X'X=\begin{bmatrix} N & n_1 & n_2 & \cdots & n_a \\ n_1 & n_1 & 0 & \cdots & 0 \\ n_2 & 0 & n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ n_a & 0 & 0 & \cdots & n_a \end{bmatrix}$$

To find a non-singular principal minor of order a, select a row and coulmn from $X'X$. Then this matrix is

$$B_{a\times a}=\begin{bmatrix} n_1 & 0 & \cdots & 0 \\ 0 & n_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & n_a \end{bmatrix},$$

then

$$B^{-1}=\begin{bmatrix} 1/n_1 & 0 & \cdots & 0 \\ 0 & 1/n_2 & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1/n_a \end{bmatrix}.$$

Thus,

$$U=(X'X)^{-}X'=\begin{bmatrix} 0 & \phi \\ \phi & B^{-1} \end{bmatrix}\begin{bmatrix} 1_N' \\ \left(\sum_{i=1}^{a}{}^{+}1_{n_i}\right)' \end{bmatrix}=\begin{bmatrix} \phi \\ \left(\sum_{i=1}^{a}{}^{+}\frac{1}{n_i}1_{n_i}\right)' \end{bmatrix}.$$

Therefore,

$$X^{+}_{(a+1)\times N}=V'XU=(1/(a+1))\begin{bmatrix} a & 1_a' \\ 1_a & (a+1)I-J \end{bmatrix}\begin{bmatrix} \phi \\ \left(\sum_{i=1}^{a}{}^{+}\frac{1}{n_i}1_{n_i}\right)' \end{bmatrix}$$

$$=\frac{1}{a+1}\begin{bmatrix} \frac{1}{n_1}1'_{n_1} & \frac{1}{n_2}1'_{n_2} & \cdots & \frac{1}{n_a}1'_{n_a} \\ \frac{a}{n_1}1'_{n_1} & -\frac{1}{n_2}1'_{n_2} & \cdots & -\frac{1}{n_a}1'_{n_a} \\ -\frac{1}{n_1}1'_{n_1} & \frac{a}{n_2}1'_{n_2} & \cdots & -\frac{1}{n_a}1'_{n_a} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{1}{n_1}1'_{n_1} & -\frac{1}{n_2}1'_{n_2} & \cdots & \frac{a}{n_a}1'_{n_a} \end{bmatrix}.$$

2. Derivation of the Moore-Penrose inverse of X for the two-way classification with no interaction and one observation per cell.

To find X^{+}, first we should find V such that $XX'V=X$.

$$(XX')_{rc\times rc}=\begin{bmatrix} I+2J & I+J & \cdots & I+J \\ I+I & J+2J & \cdots & I+J \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ I+J & I+J & \cdots & I+2J \end{bmatrix},$$

where I and J are $c\times c$ matrix.

To find a non-singular principal minor of order $r+c-1$ (=rank X), select first c row and column and first row and column from next each r row and column. Then this matrix is

$$M_{(r+c-1)\times(r+c-1)}=\begin{bmatrix} 3 & 21_1' & 21_2' \\ 21_1 & I_1+2J_1 & J_3' \\ 21_2 & J_3 & I_2+2J_2 \end{bmatrix},$$

where

1_1 is $(c-1)\times 1$ matrix,

1_2 is $(r-1)\times 1$ matrix,

I_1 is $(c-1)\times(c-1)$ matrix,

I_2 is $(r-1)\times(r-1)$ matrix,

J_1 is $(c-1)\times(c-1)$ matrix,

J_2 is $(r-1)\times(r-1)$ matrix,

and J_3 is $(r-1)\times(c-1)$ matrix.

Since M^{-1} exists, $MM^{-1}=I$ as

$$\begin{bmatrix} 3 & 21_1' & 21_2' \\ 21_1 & I_1+2J_1 & J_3' \\ 21_2 & J_3 & I_2+2J_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 1_1' & a_3 1'_2 \\ a_2 1_1 & b_2 I_1+c_2 J_1 & b_3 J_3' \\ a_3 1_2 & b_2 J_3 & d_3 I_2+e_3 I_2 \end{bmatrix} = \begin{bmatrix} 1 & \phi & \phi \\ \phi & I & \phi \\ \phi & \phi & I \end{bmatrix}$$

where the constants $a_1, a_2, a_3, b_2, b_3, c_2, d_3$ and e_3 are to be determined. Thus M^{-1} is given by

$$M^{-1}=\frac{1}{cr+r+c}\begin{bmatrix} 3cr-r-c & -2r1_1' & -2c1_2' \\ -2r1_1 & (cr+r+c)I_1-(r+1)J_1 & J_3' \\ -2c1_2 & J_3 & (cr+r+c)I_2-(c+1)J_2 \end{bmatrix}.$$

Similarly to the previous section, letX^* be the matrix that is selected the first c row in $i=1$ and a first row in $i=2$ from the matrix X_{ij}. It becomes

$$X^*=\begin{bmatrix} 1 & 1 & \phi & 1 & \phi \\ 1_1 & 1_1 & \phi & 1 & I_1 \\ 1_2 & 0 & I_2 & 1_2 & \phi \end{bmatrix},$$

then

$$(M^{-1}X^*)_{(r+c-1)\times(r+c+1)}$$

$$=\frac{1}{cr+r+c}\begin{bmatrix} -cr+r+c & cr+r-c & -2c1_2' & cr+c-r & -2r1_1' \\ r1_1 & 1_1 & J_3' & -(r+1)1_1 & (cr+r+c)I_1-(r+1)J_1 \\ c1_2 & -(c+1)1_2 & (cr+r+c)I_2-(c+1)J_2 & 1_2 & J_3 \end{bmatrix}$$

$$=V^*\,(\text{say})$$

$$V'X=V^{*\prime}X^*$$

$$=\frac{1}{cr+r+c}\begin{bmatrix} -cr+r+c & r1_1' & c1_2' \\ cr+r-c & 1_1' & -(c+1)1_2' \\ -2c1_2 & J_3 & (cr+r+c)I_2-(c+1)J_2' \\ cr+c-r & -(r+1)1_1' & 1'_2 \\ -2r1_1 & (cr+r+c)I_1-(r+1)J_1' & J_3' \end{bmatrix}\begin{bmatrix} 1 & 1 & \phi & 1 & \phi \\ 1_1 & 1_1 & \phi & 1 & I_1 \\ 1_2 & 0 & I_2 & 1_2 & \phi \end{bmatrix}$$

$$
=\frac{1}{cr+r+c}\begin{bmatrix} cr & c & c1_2' & r & r1_1' \\ c & cr+r-1 & -(c+1)1_2' & 1 & 1_1' \\ c1_2 & -(c+1)1_2 & \begin{matrix}(cr+c+r)I_2\\ -(c+1)J_2\end{matrix} & 1_2' & J_3' \\ r & 1 & 1_2 & cr+c-1 & -(r+1)1_1' \\ r1_1 & 1_1 & J_3 & -(r+1)1_1 & \begin{matrix}(cr+r+c)I_1\\ -(r+1)J_1'\end{matrix} \end{bmatrix}.
$$

Next, find U such that $X'XU=X'$.

$$
X'X_{(r+c+1)\times(r+c+1)}=\begin{bmatrix} rc & c & c1_2' & r1_3' \\ c & c & \phi & 1_3' \\ c1_2 & \phi & cI_2 & J_4' \\ r1_3 & 1_3 & J_4 & rI_3 \end{bmatrix},
$$

where 1_3 is $c\times 1$ matrix, I_3 is $c\times c$ matrix, and J_4 is $c\times(r-1)$ matrix.

To find a non-singular principal minor of order $r+c-1$, select the last $(r+c-1)$ rows and columns.

$$
B=\begin{bmatrix} cI_2 & J_3' \\ J_3 & rI_3 \end{bmatrix},\ \text{then}\ B^{-1}=\begin{bmatrix} \frac{1}{c}I_2+\frac{1}{c}J_2 & -\frac{1}{c}J_4' \\ -\frac{1}{c}J_4 & \frac{1}{r}I_3+\frac{r-1}{cr}J_5 \end{bmatrix},
$$

where J_5 is a $c\times c$ matrix, all elements of which are unity.

$$
U_{(r+c+1)\times rc}=(X'X)^{-}X'
$$

$$
=\begin{bmatrix} 0 & 0 & \phi & \phi \\ 0 & 0 & & \\ \phi & \phi & & B^{-1} \end{bmatrix}\begin{bmatrix} 1_3' & 1_3' & 1_3' & \cdots & 1_3' \\ 1_3' & \phi & \phi & \cdots & \phi \\ \phi & 1_3' & \phi & \cdots & \phi \\ \cdot & & \cdot & \cdots & \cdot \\ \phi & \phi & \phi & \cdots & 1_3' \\ I_3 & I_3 & I_3 & \cdots & I_3 \end{bmatrix}
$$

$$
=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{c}1_3' & \frac{1}{c}1_3' & \phi & \cdots & \phi & \phi \\ -\frac{1}{c}1_3' & \phi & \frac{1}{c}1_3' & \cdots & \phi & \phi \\ -\frac{1}{c}1_c' & \phi & \phi & \cdots & \phi & \frac{1}{c}1_3' \\ M & M & M & \cdots & M & M \end{bmatrix},
$$

where $M=\frac{1}{r}I_3+\frac{r-1}{rc}J_5$.

Therefore, we can get X^+ by $X^+_{(r+c+1)\times rc}=V'XU$.

$$X^+_{(r+c+1)\times(rc)}=\frac{1}{cr+c+r}\begin{bmatrix} 1_c' & 1_c' & 1_c' & \cdots & 1_c' \\ \frac{cr+r-1}{c}1_c' & -\frac{(c+1)}{c}1_c' & -\frac{(c+1)}{c}1_c' & \cdots & -\frac{(c+1)}{c}1_c' \\ -\frac{(c+1)}{c}1_c' & \frac{cr+r-1}{c}1_c' & -\frac{(c+1)}{c}1_c' & \cdots & -\frac{(c+1)}{c}1_c' \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -\frac{(c+1)}{c}1_c' & -\frac{(c+1)}{c}1_c' & -\frac{(c+1)}{c}1_c' & \cdots & \frac{cr+r-1}{c}1_c' \\ K & K & K & \cdots & K \end{bmatrix},$$

where $K=(c+2)\ I_c-\frac{r+1}{r}J_c$ and J_c is $c\times c$ matrix.

3. Derivation of the Moore-Penrose inverse of X for the two-way classification with no interaction and k observations per cell.

If we select rows which have subscription 1 in p, this matrix is the same as X of the two-way classification with one observation per cell and rank is same. Therefore, $V'X$ is same with previous section. But

$$X'X=\begin{bmatrix} rc & c & c1_2' & r1_3' \\ c & c & 0 & 1_3' \\ c1_2 & \phi & cI_2 & J_4' \\ r1_3 & 1_3 & J_4 & rI_2 \end{bmatrix}k\ ,$$

then a non-singular minor of order $r+c-1$ is $B^*=kB$. Thus $B^{*-1}=\frac{1}{k}B^{-1}$.

$$U=(X'X)^-X'=\begin{bmatrix} 0 & 0 & \phi & \phi \\ 0 & 0 & \multicolumn{2}{c}{B^{*-1}} \\ \phi & \phi & & \end{bmatrix}X$$

$$=\frac{1}{k}\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdot & \cdots & 0 \\ -\frac{1}{c}1_5' & \frac{1}{c}1_5' & \phi & \cdot & \cdots & \phi \\ -\frac{1}{c}1_5' & \phi & \frac{1}{c}1_5' & \phi & \cdots & \phi \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ -\frac{1}{c}1_5' & \phi & \phi & \cdot & \cdots & \frac{1}{c}1_5' \\ H & H & H & \cdot & \cdots & H \end{bmatrix},$$

where 1_5 is $ck\times 1$ matrix,

$$H=\begin{bmatrix} \frac{c+r-1}{rc}1_3' & \frac{r-1}{rc}1_3' & \cdots & \frac{r-1}{rc}1_3' \\ \frac{r-1}{rc}1_3' & \frac{c+r-1}{rc}1_3' & \cdots & \frac{r-1}{rc}1_3' \\ \cdot & \cdot & \cdots & \cdot \\ \frac{r-1}{rc}1_3' & \frac{r-1}{rc}1_3' & \cdots & \frac{c+r-1}{rc}1_3' \end{bmatrix}.$$

Therefore,

$X^+=V'XU$

$$=\begin{bmatrix} 1_5' & 1_5' & 1_5' \cdots & 1_5' \\ \frac{cr+r-1}{c}1_5' & -\frac{c+1}{c}1_5' & \cdot \cdots & -\frac{c+1}{c}1_5' \\ -\frac{c+1}{c}1_5' & \frac{cr+r-1}{c}1_5' & \cdot \cdots & -\frac{c+1}{c}1_5' \\ \cdot & \cdot & \cdot \cdots & \cdot \\ -\frac{c+1}{c}1_5' & -\frac{c+1}{c}1_5' & \cdot \cdots & \frac{cr+r-1}{c}1_5' \\ H_1 & H_1 & \cdot \cdots & H_1 \end{bmatrix}$$

where

$$H_1=\begin{bmatrix} \frac{cr+r-1}{r}1_3' & -\frac{r-1}{r}1_3' & \cdots & -\frac{r-1}{r}1_3' \\ -\frac{r-1}{r}1_3' & \frac{cr+r-1}{r}1_3' & \cdots & -\frac{r-1}{r}1_3' \\ \cdot & \cdot & \cdots & \cdot \\ -\frac{r-1}{r}1_3' & -\frac{r-1}{r}1_3' & \cdots & \frac{rc+r-1}{r}1_3' \end{bmatrix}$$

4. Derivation of the Moore-Penrose inverse of X for the two-way classification with interaction and one observation per cell.

To get X^+, first find $(X'X)^-$. Since the rank of $X'X$ is rc, so this matrix is

$$(X'X)^-=\begin{bmatrix} 0 & 0 \\ 0 & I_{rc\times rc} \end{bmatrix}.$$

Now we find projection operator Px, where $Px=X(X'X)^-X'$. It can be shown easily that $Px=I_{rc\times rc}$. We know that $X^+=V'Px$, where $V=(XX')^-X$. Consequently, X^+ is the same as V'. Note that (XX') has the following form

$$(XX')=\begin{bmatrix} 2I+2J & I+J & \cdots & \cdots & I+J \\ I+J & 2I+2J & I+J & \cdots & I+J \\ I+J & I+J & 2I+2J & \cdots & I+J \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ I+J & I+J & \cdots & \cdots & 2I+2J \end{bmatrix}$$

where I and J are $c\times c$ matrices, XX' is $rc\times rc$ matrix with $r>1$. It is simple to find a generalized inverse of XX'. Since rank $(XX')=rc$, so $(XX')^{-1}$ exists, and the result is as follows

$$(XX')^{-}=(XX')^{-1}=\begin{bmatrix} aI+bJ & cI+dJ & \cdots & cI+dJ \\ cI+dJ & aI+bJ & \cdots & cI+dJ \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ cI+dJ & cI+dJ & \cdots & aI+bJ \end{bmatrix},$$

where $a=r/(r+1)$, $b=-r/(r+1)(c+1)$, $c=-1/(r+1)$, $d=1/(r+1)(c+1)$ and I,J are $c\times c$ matrices. Thus X^{+} has the general form of Result 3.4.1.

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