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Optimal Run Orders in Factorial Designs

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ABSTRACT

It is often necessary to obtain some run orders in factorial designs which have a small number of factor level changes and a small linear time trend. In this paper we propose an algorithm to find optimal or near-optimal run orders for 2^4, 2^5, 3^2 and $2 \cdot 3^2$ factorial designs under the criterion that the number of factor level changes and the linear time trend should be simultaneously small.

1. Introduction

Draper and Stoneman(1968) examined eight-run two level factorial and fractional factorial designs when it is physically difficult to change levels of factors, and there may be some significant linear time trend. Later Dickinson(1974) studied run orders for the 2^4 and 2^5 designs requiring a minimum number of factor level changes and a small linear time trend.

The problem of time trends is frequently present in many cases. Traditionally, the elimination of time trends has been accomplished by use of blocking. These techniques work satisfactorily in many cases, but, there are situations in which these techniques are inadequate to provide all of the desired information. Experimental designs which can be used in these situations have been developed by several workers.

Box(1952, 1958) and Cox(1951, 1952, 1958) proposed the experimental designs which allow adjustment for time trends. Hill(1960) combined the designs of Box and Cox to form new designs which allow one to study factors in the presence of a time trend. A

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paper by Daniel and Wilcoxon (1966) studied the adverse effects of linear and quadratic time trends on the estimated effects arising from various two-level factorial designs.

In this paper we propose an algorithm to find optimal or near-optimal run orders for $2^4, 2^5, 3^2$ and $2\cdot 3^2$ factorial designs which have a small number of factor level changes and a small linear time trend. However, the basic concept of the proposed algorithm can be used to find good run orders for any types of $2^n\cdot 3^m$ factorial designs when it is desirable to obtain small number of factor level changes and small linear time trend.

2. Background of Run Orders

Table 1 shows the 2^3 design with the run orders in "the standard order". Suppose that there is a linear time trend. We can simulate this by setting observations $y_1, y_2, \cdots, y_8$ equal to $1, 2, \cdots, 8$, respectively. The effect of the linear time trend can be obtained by taking the inner product of the effect column with the observation column. This results in a time count of 4, 8 and 16 for the three effects A, B and C, respectively. Thus, we conclude that if the 2^3 design is run in the standard order, 11 factor level changes are required and the maximum time count is 16.

Draper and Stoneman (1968) reported that the run orders 14865732 as shown in Table 2-1 is considered to be the best ordering, since it has a small number of factor level changes and a small maximum time count.

For 2^4 design, Dickinson (1974) used a search technique to generate all possible 238

Table 1. The 2^3 design in the standard order

Run number	Effect			Observation
	A	B	C	
1	−	−	−	y_1
2	+	−	−	y_2
3	−	+	−	y_3
4	+	+	−	y_4
5	−	−	+	y_5
6	+	−	+	y_6
7	−	+	+	y_7
8	+	+	+	y_8
number of factor level changes	7	3	1	total = 11
time count	4	8	16	maximum time count = 16

Table 2-1. "best" order by Draper and Stoneman

Run number	A	B	C	
1	−	−	−	
4	+	+	−	
8	+	+	+	
6	+	−	+	
5	−	−	+	
7	−	+	+	
3	−	+	−	
2	+	−	−	
number of factor level changes	3	4	2	total=9
time count	−2	0	0	maximum time count=2

Table 2-2. "best" order for 2^3 design by Dickinson

Run number	A	B	C	
1	−	−	−	
2	+	−	−	
6	+	−	+	
8	+	+	+	
4	+	+	−	
3	−	+	−	
7	−	+	+	
5	−	−	+	
number of factor level changes	2	2	3	total=7
time count	−8	8	8	maximum time count=8

fundamentally different run orders in which the number of foctor level changes was held to the minimum number of 15. Note that a set of equivalent run orders (which are not fundamentally different run orders) can be obtained by permuting columns and/or reversing the signs of the elements in one or more columns of a design. He chose the "best" order that has the least maximum time count among 238 fundamentally different run orders. This order and the standard order are shown in Tables 3-1 and 3-2. Note that for 2^3 design the "best" order by Dickinson is different from that of Draper and Stoneman as compared in Tables 2-1 and 2-2.

Table 3-1. Standard order for 2^4 design

Run number	A	B	C	D	
1	−	−	−	−	
2	+	−	−	−	
3	−	+	−	−	
4	+	+	−	−	
5	−	−	+	−	
6	+	−	+	−	
7	−	+	+	−	
8	+	+	+	−	
9	−	−	−	+	
10	+	−	−	+	
11	−	+	−	+	
12	+	+	−	+	
13	−	−	+	+	
14	+	−	+	+	
15	−	+	+	+	
16	+	+	+	+	
number of factor level changes	15	7	3	1	total = 26
time count	8	16	32	64	maximum time count = 64

Table 3-2. "best" order for 2^4 design by Dickinson

Run sequence	Run number in standard order	A	B	C	D	
1	1	−	−	−	−	
2	2	+	−	−	−	
3	10	+	−	−	+	
4	14	+	−	+	+	
5	16	+	+	+	+	
6	15	−	+	+	+	
7	7	−	+	+	−	
8	3	−	+	−	−	
9	11	−	+	−	+	
10	12	+	+	−	+	
11	4	+	+	−	−	
12	8	+	+	+	−	
13	6	+	−	+	−	
14	5	−	−	+	−	
15	13	−	−	+	+	
16	9	−	−	−	+	
number of factor level changes		4	2	4	5	total = 15
time count		−16	0	16	0	maximum time count = 16

3. Proposition of a Search Technique

Consider the following six columns (a), (b), (c), (a′), (b′) and (c′).

(a)	(b)	(c)
−	−	−
+	−	−
+	−	+
+	+	+
+	+	−
−	+	−
−	+	+
−	−	+
(a′)	**(b′)**	**(c′)**
−	−	+
−	+	+
−	+	−
+	+	−
+	+	+
+	−	+
+	−	−
−	−	−

Here, (a), (b) and (c) are the respective columns for the factors A, B and C in the order which has a minimum number of factor level changes in 2^3 design. (a′), (b′) and (c′) are obtained by reversing the run sequence of (a), (b) and (c), respectively. Therefore, (a′), (b′) and (c′) are symmetric with the columns (a), (b) and (c), respectively.

Figure 1. A 2^4 arrangement when (4) and (8) are columns of 8 elements of +, − signs

Any order which has a minimum number of factor level changes in 2^3 design (a), (b), (c)	(4)
The order which is symmetric with the above order (a′), (b′), (c′)	(8)

Consider a 2^4 arrangement in Figure 1. There are 2^8 possible arrangements in the column (4). To avoid the repetition of same runs, the column (8) is uniquely determined by the columns (a), (b), (c), (4), (a′), (b′) and (c′). Therefore, $256(=2^8)$ possible arrangements are all that we have to conisder.

Our claim is that the best order among these 256 arragements is the best or at least nearly best order among all the possible 16! arrangements. This claim is based on the following facts. The combination of the columns (a), (b) and (c) is the order which has a minimum number of factor level changes of 7 in 2^3 design and the columns (a′), (b′) and (c′) have the same minimum number of level changes. Also, since (a′), (b′) and (c′) are symmetric with (a), (b) and (c), the combination of these 6 columns provides zero maximum time count.

Of all the 2^8 arrangements in the columns (4) and (8), we want to find the order of signs which has a small maximum time count and a small number of factor level changes. To support this technique, we apply this idea to obtain the best order of a 2^3 design. One of the run orders which has a minimum number of factor level changes of 3 in 2^2 design is the run sequence 1243 as follows.

Run number	A	B
1	−	−
2	+	−
4	+	+
3	−	+

Using the above technique, we have $16(=2^4)$ possible arrangemntes to consider. The best order among these 16 arrangements is 16873425 as follows.

Run number	A	B	C
1	−	−	−
6	+	−	+
8	+	+	+
7	−	+	+
3	−	+	−
4	+	+	−
2	+	−	−
5	−	−	+

Note that the bottom column of $C(-,-,-,+)$ is uniquely determined to avoid the repetition of runs from the results of A, B and the upper column of C. For instance, the run number 3 should have $(-, +, -)$ signs, since the run number 7 already has the sign $(-, +, +)$.

The best order of 14865732 in Table 2-1 can be obtained from 16873425 by exchanging A, B, C columns. Therefore, they are equivalent run orders from two points of view such as maximum time count and number of factor level changes.

Let us apply this technique to obtain the best order of a 2^4 design. Table 4 shows the number of level changes and the maximum linear time count.

Table 4. Classification of 2^4 design run orders by level changes and time count.

Number of factor level changes	Number of run orders	Number of run orders with maximum time count								
		0	2	4	6	8	10	12	……	64
15	2	0	0	0	0	0	0	0		
17	14	0	0	0	0	2	0	0		
19	42	2	2	0	2	0	2	0		
21	70	0	6	2	4	4	4	2		
23	70	2	2	4	4	8	4	4		
25	42	2	4	4	4	0	2	6		
27	14	2	0	4	0	0	2	0		
29	2	0	0	0	0	2	0	0		
Total	256	8	14	14	14	16	14	12	……	12

We see from Table 4 that two run orders which lie above the boundary are better than the rest from the points of view we have considered. These two run orders are shown in Table 5 in which double-digit order numbers are distinguished by a parenthesis. Note that the second one is nothing but the reverse order of the first one. Hence, we can say that the first one is the "best" order. Table 6 gives this sequence.

Table 5. Two desirable run orders for 2^4 design

order of runs	Number of factor leval changes	Maximum time count
12(14)(16)(12)(11)75(13)(15)3486(10)9	19	0
9(10)6843(15)(13)57(11)(12)(16)(14)21	19	0

We want to verify this claim by making 300,000 arrangements which are randomly chosen among $16!\,(=2.092\times10^{13})$ arrangements by computer. The classification of these 300,000 arrangements is given in Table 7.

Table 6. The best order of a 2^4 deiign

Run sequence	Run number	A	B	C	D	
1	1	−	−	−	−	
2	2	+	−	−	−	
3	14	+	−	+	+	
4	16	+	+	+	+	
5	12	+	+	−	+	
6	11	−	+	−	+	
7	7	−	+	+	−	
8	5	−	−	+	−	
9	13	−	−	+	+	
10	15	−	+	+	+	
11	3	−	+	−	−	
12	4	+	+	−	−	
13	8	+	+	+	−	
14	6	+	−	+	−	
15	10	+	−	−	+	
16	9	−	−	−	+	
number of factor level changes		4	4	6	5	total=19
time count		0	0	0	0	maximum time count=0

Table 7. Classification of 300,000 randomly chosen run orders of 2^4 design by factor level changes and time count

Number of factor level changes	Number of run orders	Number of run orders with maximum time count											
		0	2	4	6	8	10	12	14	16	18	20	…
17	3	0	0	0	0	0	0	0	0	0	0	0	
18	3	0	0	0	0	0	0	0	0	0	0	0	
19	18	0	0	0	0	0	0	0	0	0	0	1	
20	33	0	0	0	0	0	0	0	0	0	0	0	
21	142	0	0	0	0	0	0	1	1	4	8	16	
22	251	0	0	0	0	2	4	2	8	7	7	18	
23	1,164	0	0	0	1	2	2	8	6	6	14	29	
24	1,826	0	0	1	3	4	7	8	18	14	22	37	
25	5,218	0	0	0	5	7	9	15	23	27	56	80	
26	6,178	0	2	4	11	12	29	37	56	80	95	114	
⋮													
49													
Total	300,000	42	352	943	2,451	4,230	6,743	9,392	12,634	15,333	19,604	21,321	…

Comparing Table 4 with Table 7 we can give a rough verification to our claim that the best order among the 256 arragements is the best or at least nearly best order among all the possible 16! arrangements. The best order given by Dickinson in Table 3-2 has 15 factor level changes and maximum time count 16. If the main effects of factors A, B and C are significantly affected by the linear time trend, the best order given by this proposed technique could be better than the best order given by Dickinson.

4. Extension to the 2^5 Design

In Dickinson's paper(1974) he applied his search algorithm to find the best order in 2^5 design. However, the number of fundamentally different run orders possessing the minimum number of 31 factor level changes for the 2^5 design is very large. He mentioned that "Exhaustive examination of even this limited subset of the run orders for the 2^5 design is clearly infeasible." Hence, he considered stratified sampling and random sampling of the run orders with 31 factor level changes. He obtained 4 run orders with a maximum time count of 36.

On the other hand, the application of the propose technique introduced in the previous section, to the 2^5 design can considerably reduce the computational burden. The starting order of runs which has the minimu number of level changes of 15 in 2^4 design is 12 (10)(14)…5(13)9 which was presented in Table 3—2. This statarting order can be also obtained in the process of classification of 2^4 design run orders presented in Table 4. In the proposed technique we have only to consider 2^{16}(=65,536) arrangements, which are clearly feasible to examine by computers.

Table 8 shows the tabulation of the number of factor level changes and the maximum

Table 8. Classification of 2^5 design run orders by factor level changes and time count

Number of factor level changes	Number of run orders	Number of run orders with maximum time count											
		0	2	4	6	8	9	12	14	16	18	…	256
31	2	0	0	0	0	0	0	0	0	0	0		0
33	30	0	0	0	0	0	0	0	2	0	0		
35	210	2	0	0	2	2	2	0	2	4	2		
37	910	2	18	6	8	14	12	4	8	14	10		
39	2,730	26	30	28	32	42	32	20	30	56	28		
41	6,006	54	70	92	80	96	80	88	94	84	76		
⋮													
61	2	0	0	0	0	0	0	0	0	2	0		
Total	65,536	695	1,374	1,368	1,370	1,376	1,364	1,352	1,354	1,356	1,338	…	2

time count. The "best" order among these 2^{16} arrangements turns out to be 12(10)(14)(32)⋯(26)(18)(17) as shown in Table 9.

Table 9. The best order of 2^5 design by the proposed technique

Run sequence	Run number	A	B	C	D	E	
1	1	−	−	−	−	−	
2	2	+	−	−	−	−	
3	10	+	−	−	+	−	
4	14	+	−	+	+	−	
5	32	+	+	+	+	+	
6	31	−	+	+	+	+	
7	23	−	+	+	−	+	
8	19	−	+	−	−	+	
9	27	−	+	−	+	+	
10	28	+	+	−	+	+	
11	20	+	+	−	−	+	
12	24	+	+	+	−	+	
13	6	+	−	+	−	−	
14	5	−	−	+	−	−	
15	13	−	−	+	+	−	
16	9	−	−	−	+	−	
17	25	−	−	−	+	+	
18	29	−	−	+	+	+	
19	21	−	−	+	−	+	
20	22	+	−	+	−	+	
21	8	+	+	+	−	−	
22	4	+	+	−	−	−	
23	12	+	+	−	+	−	
24	11	−	+	−	+	−	
25	3	−	+	−	−	−	
26	7	−	+	+	−	−	
27	15	−	+	+	+	−	
28	16	+	+	+	+	−	
29	30	+	−	+	+	+	
30	26	+	−	−	+	+	
31	18	+	−	−	−	+	
32	17	−	−	−	−	+	
number of factor level changes		8	4	8	10	5	total=35
time count		0	0	0	0	0	maximum time count=0

5. Extension to Larger 2^n Designs

Application of the proposed method to 2^n designs where n is greater than 5 becomes computationally difficult, since we need to consider $2^{(2^{n-1})}$ arrangements. However, we can suggest that, based on the philosophy of the proposed technique, the following order of runs in Figure 2 is a very good order of runs which possesses a small number of factor level changes and zero maximum time count.

Figure 2. A good choice of run orders for 2^n designs

Any order which has a minimum number of factor level changes in 2^{n-1} design	$-$ $-$ $\cdot$ $\}\ 2^{n-3}$ $\cdot$ $-$ $+$ $+$ $\cdot$ $\}\ 2^{n-2}$ $\cdot$ $+$ $-$ $-$ $\cdot$ $\}\ 2^{n-3}$ $\cdot$ $-$
The order of signs which is symmetric with the above order	$+$ $+$ $\cdot$ $\}\ 2^{n-3}$ $\cdot$ $+$ $-$ $-$ $\cdot$ $\}\ 2^{n-2}$ $\cdot$ $-$ $+$ $+$ $\cdot$ $\}\ 2^{n-3}$ $\cdot$ $+$

Consider the last column of Figure 2. This column has 5 factor level changes and its linear time count is zero. Therefore, the order of runs in Figure 2 has zero maximum time count, and the total number of factor level changes is $2\times$(the minimum number of factor level changes in 2^{n-1} design)$+5=2\times(2^{n-1}-1)+5=2^n+3$, which is relatively very small. Note that an order which has a minimum number of factor level changes in 2^{n-1} design can be obtained in such a way that the first run is $(-, -, \cdots, -)$ and the subsequent runs have the signs which are different in only one column compared with the previous run. Also note that by this proposed method, the number of factor level changes is only 4 more than the minimum 2^{n-1}, which becomes insignificant as n gets large.

6. 3^2 Design

Table 10 shows a 3^2 design with runs in the standard order. Suppose there is a linear time trend. We can simulate this by setting observations $y_1, y_2, \cdots, y_9$ equal to $1, 2, \cdots, 9$, respectively. The effect of the linear time trend can be obtained by the same method as the 2^3 design. The 9! arrangements divide into 45,360 distinct sets of 8 equivalent run orders per set. The equivalent run orders can be obtained from one another by permutation of factor numbers(2!) and by switching the signs of each column(2^2). Hence, $2! \times 2^2 = 8$.

Table 10. 3^2 design with the runs in the standerd order

Run number	A	B	observation
1	−1	−1	y_1
2	0	−1	y_2
3	1	−1	y_3
4	−1	0	y_4
5	0	0	y_5
6	1	0	y_6
7	−1	1	y_7
8	0	1	y_8
9	1	1	y_9
number of factor level changes	8	2	total=10
time count	6	18	maximum time count=18

The complete classification is shown in Table 11. Table 12 shows the two run orders which lie above the boundary shown in the top left hand corner of Table 11.

7. $2 \cdot 3^2$ Design

The direct application of the Draper and Stoneman's method to the $2 \cdot 3^2$ design is formi-

Table 11. Classification of 3^2 design run orders by factor level changes and time count

Number of factor level changes	Number of run orders	Number of distinct run orders	Number of distinct run orders with maximum time count														
			0	1	2	3	4	5	6	7	8	9	10	11	12	…	18
8	1,512	189	2	0	2	12	6	14	8	4	6	40	2	0	27	…	12
9	11,376	1,422	0	16	20	64	32										
10	38,744	4,968	4	58	77												
11	77,328	9,666	8	176													
12	102,960	12,870	20														
13	77,328	9,666	22														
14	39,744	4,968	33														
15	11,376	1,422	6														
16	1,512	189	0														
Total	362,880	450,360	95	1,084	1,915	2,990	3,360	4,228	4,308	4,712	4,197	3,824	3,301	3,288	2,552		108

Table 12. Two desirable arrangements of run orders in 3^2 design

Order of runs	Number of factor level changes	Maximum time count
139874652	8	0
254698713	8	0

dable in terms of computing time because of the large number of possible arrangements ($18!=6.4\times10^{15}$). Table 13 shows a $2\cdot3^2$ design with run orders in the standard order (A, B : three-level, C : two-level).

We can simulate the linear time trend by setting observations $y_1, y_2, \cdots, y_{18}$ equal to $1, 2, \cdots, 18$, respectively. It is possible to apply the proposed method in this paper to the $2\cdot3^2$ design by starting with any order which has the minimum number of factor level changes of 8 in 3^2 design. Using this technique, we have only to consider 2^9 ($=512$) arrangements. The classification of these 512 arrangements is shown in Table 14. Table 15 shows the two run orders which lie above the boundary indicated in the top left hand corner of Table 14.

8. Concluding Remarks

We have considered the determination of run orders in $2^4, 2^5, 3^2$ and $2\cdot3^2$ designs when it is difficult to change factor levels and linear time trends exist. With modern computing facilities, it would seem infeasible to consider all possible arrangements for $2^n (n\geq4)$

Table 13. $2\cdot3^2$ design with run orders in the standard order

Run number	A	B	C	observation
1	−1	−1	−1	y_1
2	0	−1	−1	y_2
3	1	−1	−1	y_3
4	−1	0	−1	y_4
5	0	0	−1	y_5
6	1	0	−1	y_6
7	−1	1	−1	y_7
8	0	1	−1	y_8
9	1	1	−1	y_9
10	−1	−1	1	y_{10}
11	0	−1	1	y_{11}
12	1	−1	1	y_{12}
13	−1	0	1	y_{13}
14	0	0	1	y_{14}
15	1	0	1	y_{15}
16	−1	1	1	y_{16}
17	0	1	1	y_{17}
18	1	1	1	y_{18}
number of factor level changes	17	5	1	total=23
time count	12	36	81	maximum time count=81

Table 14. Classification of $2\cdot3^2$ design run orders by factor level changes and time count

Number of factor level chenges	Number of run orders	Number of run orders with maximum time count								
		1	3	5	7	9	11	13	…	81
17	2	0	0	0	0	0	0	0		
19	16	0	0	0	0	2	0	0		
21	56	2	2	0	0	2	2	0		
23	112	8	2	4	8	0	2	2		
25	140	4	8	6	8	8	6	6		
27	112	4	8	4	6	8	12	4		
29	56	6	2	8	0	4	0	8		
31	16	2	2	2	2	0	0	2		
33	2	0	0	0	0	2	0	0		
Total	512	26	24	24	24	26	22	22	…	2

and $2\cdot3^2$ designs. In the proposed method of this paper, we had only to consider a small portion of all possible arrangements. However, we could find optimal or near-optimal run orders for 2^4, 2^5, 3^2 and $2\cdot3^2$ designs under the criterion that the number of factor

Table 15 Two desirable run orders for $2 \cdot 3^2$ design

Run orders	Number of factor level changes	Maximum time count
13(18)(17)(16)(13)652(11)(14)(15)4789(12)(10)	21	1
(10)(12)(18)874652(11)(14)(15)(13)(16)(17)931	21	1

level changes and the maximum time count should be simultaneously small.

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