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On the Output of Two-Stage Cyclic Queue

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Abstract

Throughout this paper we analyze the system at output point t of two stage cyclic queueing model. Our main result characterize the stochastic process (X°, T°), the system at output point, as a Markov renewal process. The subsequent lemma exhibits the semi-Markov kernel of (X°, T°) with state dependent feedback, the possibility of a reducible state space arises. A simple necessary and sufficient condition for the irreducibility of (X°, T°) was determinded. This irreducibility implied that (X°, T°) was aperiodic.

1. Introduction

The concept of a cyclic queue has been introduced by Koenisberg (1958). P. D. Finch (1959) described cyclic queue with feedback in which units pass in turn series of servers to return to the initial server; feedback is permitted either by feedback from the terminal server or by feedback from each server of the series to the queue waiting for service at that stage. Two queues in tandem attended by a single server is considered by Miguel Taube Netto (1977). G.R. Davigon (1974) researched the queueing system with feedback where the service at Q_1 is over, the customer instantaneously rejoins the Q_1 queue, or leaves at Q_1 forever.

As in figure 1 an arriving customer waits for service at Q_1. When the service at Q_1 is completed, the customer either leaves (departs) the system with probability p or queues (waits) for the service at Q_2 with probability 1-p. When the service is over at Q_2, the customer rejoins the Q_1 queue. In this model both queues are assumed to have infinite capacity. Two stage cyclic queue in M/M/1 system is now applying in the computer system (see Allen). In this paper we study the output process properties in two stage cyclic queue where arrival distribution is Poisson process with average arrival rate and service distribution is G(X) at Q_1 and $1\text{-}e^{-\mu x}, \mu > 0$ at Q_2.

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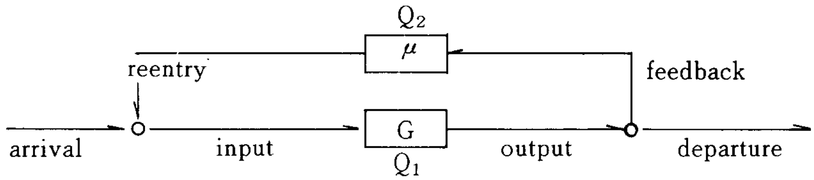

Fig. 1. Two stage cyclic queuing model

2. Method

Let $Z = \{Z_t\}$ be the time dependent queue length, S_n be the length of the n-th service, T_n° be the time of the n-th output and X_n° be an ordered triplet consisting of the two queue lengths at time T_n° and an indicator of whether the customer departs or feeds back (waits at Q_2).
Then we denote this by $P\{Z_t=(j_1,j_2) \mid X_0^\circ=(i_1,i_2,i_3),\ T_0^\circ=0\}$.
The feedback mechanism depends only on the type of the unit. Whenever there is feedback, the unit joins the Q_2 queue, we define the random variable Y_n with the n-th output;

$$Y_{n+1} = \begin{cases} 0 & \text{if the n-th output departs} \\ 1 & \text{if the n-th output feeds back} \end{cases}$$

Here the sequence $\{Y_n\}$ is called the switching process and is denoted by

$$P_r\{Y_{n+1}=k \mid Z_{T_{n+1}^\circ}=(j_1,j_2),\ S_{n+1}^{(1)}=x\}$$
$$=P\{Y_{n+1}=k \mid Z_t;\ t \le T_{n+1}^\circ\} = h_k(j_1,j_2,x) \quad \text{for } k=0,1;\ n=1,2,\cdots\ j_1,\ j_2=1,2,\cdots$$

3. Output Process

In much of our analysis we will be examining the system at output points. So this paper develops the main properties of the output process. Hence X_n° is the state of the system at time T_n° of the n-th output. X_n° is an ordered triplet. If $X_n^\circ=(i_1,i_2,i_3)$, then at time T_n°, i_1 customers are in Q_1, i_2 customers are in Q_2 and i_3 is 0 or 1 depending on whether n-th output departs of queues at Q_2 (feeds back) respectively. Also let $X^\circ=\{X_n^\circ\}$. (X°,T°) will denote $\{(X_n^\circ,T_n^\circ)\}$. We will need to establish the structure of (X°,T°).

Proposition 3.1. The random variable X_{n+1}° depends only on X_n°.
Suppose we have defined, for each $n \in N$, a random variable X_n taking values in a countable set E, $E=N \times N \times \{0,1\}$, $N=\{0,1,2,\cdots\}$ and a random variable T_n taking value in $R_+=[0,\infty)$ such that $0=T_0 \le T_1 \le T_2 \le \cdots$ Then we have next theorem.

Theorem 3.2. The stochastic process (X°,T°) is a Markov renewal process with state space $E=N \times N \times \{0,1\}$.

Proof. Whenever the n-th idle period I_{n+1} exists, the output interval $O_{n+1}=T_{n+1}^\circ - T_n^\circ$ can be written as

$$O_{n+1} = \begin{cases} S_{n+1}^{(1)} + I_{n+1} & \text{if } i_1=0, \\ S_{n+1}^{(1)} & \text{if } i_1>0. \end{cases}$$

So, if $i_1>0$, $O_{n+1}=S_{n+1}^{(1)}$. If $i_1=0$, then $O_{n+1}+I$.

If $i_2+i_3=0$, I is an exponential random variable with parameter λ.
If $i_2+i_3>0$, I is an exponential random variable with parameter $(\lambda+\mu)$
Now the number of arrivals to Q_1 and number of departures from Q_2 depends on X°_n and O_{n+1}
Thus X°_{n+1} depends only on j_1, j_2 and $S^{(1)}_{n+1}$,

$$P\{X^\circ_{n+1}=(j_1, j_2, j_3), T^\circ_{n+1}-T^\circ_n\leqq t \mid X^\circ_o,\dots, X^\circ_n, T^\circ_o,\dots,T^\circ_n\}$$
$$=P\{X^\circ_{n+1}=(j_1, j_2, j_3), T^\circ_{n+1}-T^\circ_n\leqq t \mid X^\circ_n=(i_1, i_2, i_3)\}$$

for all $(j_1, j_2, j_3)\in E$, $(i_1, i_2, i_3)\in E$, $n\in N$ and $t\geq o$.

Hence (X°, T°) has the desired property.

Let the probability $Q=\{Q(i, j, t); i, j\in E, t\in R\}$ denotes the semi-Markov kernel over E, that is

$$Q_{ij}(t)=P\{X^\circ_{n+1}=j, T^\circ_{n+1}-T^\circ_n\leq t \mid X^\circ_n=i\}$$

where $i=(i_1, i_2, i_3)\in E$, $j=(j_1, j_2, j_3)\in E$.

Lemma 3.3 The transition probabilities for the Markov renewal process $\{(X^\circ_n, T^\circ_n)\}$ are given by

$$Q_{ij}(t)=P\{X^\circ_{nn}=j, T^\circ_{n+1}-T^\circ_n\leq t \mid X^\circ_n=i\} \qquad (1)$$

where

$$Q_{ij}(t)=\begin{cases}\int_0^t h_{j_3}(j_1, j_2, x)\, M^{i_2}_\mu(i_2-j_2, x)\, M^\infty_\lambda(j_1-i_1+1+j_2-i_2, x)\, dG(x), \\ \qquad \text{if } i_1>0,\ i_3=0,\ j_1-i_1+1\geqq i_2-j_2\geqq 0; \\ \int_0^t \lambda e^{-\lambda s}\, Q_{mj}(t-s)\, ds,\ m=(1,\ 0,\ 0) \\ \qquad \text{if } i_1=0,\ i_2=0,\ i_3=0,\ j_1\geqq 0,\ j_2=0, \\ \int_0^t (\lambda+\mu)\, e^{-(\lambda+\mu)s}\left(\frac{\lambda}{\lambda+\mu}Q_{mj}(t-s)+\frac{\mu}{\lambda+\mu}Q_{nj}(t-s)\, ds\right. \\ \qquad m=(1, i_2,\ 0),\ n=(1,\ i_2-1, 0) \\ \qquad \text{if } i_1=0,\ i_2>0,\ i_3=0 \\ Q_{ml}(t), m=(i_1,\ i_2+1, 0), \\ \qquad \text{if } i_1=0,\ i_2=0,\ i_3=1 \\ 0 \text{ otherwise}\end{cases}$$

and

$$M^k_\mu(j, x)=\begin{cases} e^{-\mu x}(\mu x)^i/j! & j<k, \\ \sum_{i=k}^{\infty} e^{-\mu x}(\mu x)^i/i! & j=k, \\ 0 \text{ otherwise}\end{cases}$$

Proof. From theorem 3.2., there are four cases to consider;
Case 1. $i_1>0$, $i_3=0$; from theorem 3.2 and using feedback mechanism one finds $h_{j_3}(j_1, j_2, x)$. Since $i_1>0$, $O_{n+1}=S^{(1)}_{n+1}$. Hence it has service distribution $G(x)$. i_2-j_2 customers must leave Q_2. The probability of this event occuring in time is $M^{i_2}_\mu(i_2-j_2, x)$. j_1-i_1+1 customers must enter Q_1. Also since i_2-j_2 customers come from Q_2 (reentry Q_1), $(j_1-i_1+1)-(i_2-j_2)$ must arrive from the outside, $M^\infty_\lambda(j_1-i_1+1+j_2-i_2, x)$ is the probability of this event occuring in time x and $j_1-i_1+1\geq i_2-j_2\geq 0$.
Case 2. $i_2+i_3=0$ $(i_1=i_2=i_3=0)$: this case corresponds to the situation where the system is empty. Therefore an unit must arrive to initiate the service periods, i.e., $\lambda e^{-\lambda s}$. This unit will also be the output. The probability of new transfering from (1, 0, 0) to j in time $t-s$ is given

case 1.

Case 3. $i_1=0$, $i_2>0$, $i_3=0$: Since $i_1=0$ the system waits for an exponential length of time s until either an arrival occurs or customer departs Q_2; $(\lambda+\mu)e^{-(\lambda+\mu)s}$. An arrival occurs first with probability $\lambda/(\lambda+\mu)$ and then the probability of transfering from $(1, i_2, 0)$ to j is given in case 1. Similarly an output from Q_2 occurs first with probability $\mu/(\lambda+\mu)$ and then the probability of transfering from $(1, i_2-1, 0)$ to j is given in case 1;

$$\frac{\lambda}{\lambda+\mu} Q_{mj}(t-s)+\frac{\mu}{\lambda+\mu} Q_{nj}(t-s).$$

Case 4. $i_1=0$, $i_2=0$, $i_3=1$;the probability of transfering from $(i_1, i_2, 1)$ to j is the same as the probability of transfering from $(i_1, i_2+1, 0)$ to j.

For each pair (i, j) the function $t \to Q(i, j, t)$ has all the properties of a distribution function except that

$$P(i, j)=\lim_{t\to\infty} Q(i, j, t)$$

is not necessarily one. Indeed, it is easy to see from (1) that

$$P(i, j)\geq 0, \sum_{j\in E} P(i, j)=1;$$

that is, the P(i, j) are the transition probabilities for some Markov chain with state space E.

Proposition 3.4 The stochastic process (X°, T°) is an irreducible Markov
(Criterion)
renewal process with state space E if and only if

$$0<\int_0^\infty h_0(j_1, j_2, x)\, dG(x)<1, \quad j_1, j_2 \in N \tag{2}$$

Proof. If $\int_0^\infty h_0(j_1, j_2, x)\, dG(x) = 0$ then the state $(j_1, j_2, 0)$ is not reachable.

Also if $\int_0^\infty h_0(j_1, j_2, x)\, dG(x) = 1$ then the state $(j_1, j_2, 1)$ is not reachable.

If (2) holds, then there is a positive probability of transfering from $(i_1, i_2, 0)$ to $(i_1+1, i_2\ 0)$, $(i_1-1, i_2, 0)$ or $(i_1, i_2-1, 0)$ is one step.

Similarly there is a positive probability of transferring from $(i_1, i_2, 1)$ to $(i_1, i_2+1, 0)$ or $(i_1, i_2, 0)$. By using a finite number of these steps, there exists a positive probability of reaching any state from any other state.

Theorem 3.5 The stochastic process (X°, T°) is aperiodic, if irreducible

Proof. First, some state $(0,\cdot,\cdot)$ is reachable, since in any state $(i_1,\cdot,\cdot)$, $i_1>0$, there is a positive probability of entering some state $(i_1-1,\cdot,\cdot)$. Hence we need only show that the distribution of the time between returns to $(0, i_2, i_3)$ is aperiodic. But this follows since the inter-return time is composed of a random number of service times and at least one idle time. The idle time is exponentially distributed, hence the inter-return time cannot be compute with a fixed span δ.

4. Comments

The above theorems enables us to find the stationary probability for the type of output and the stationary distribution of output interval lengths.

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