論　文

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비선형 구조해석에서 부분구조를 이용한 수정 BFGS법

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A Modified BFGS Method with Substructuring for the Nonlinear Structural Analysis

by

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Abstract

The basic BFGS procedure for the nonlinear finite element analysis is reviewed. Through a simple numerical example, promising characteristics of the method is evaluated and discussed. Based on the discussion of computational performance, a modified BFGS algorithm with substructuring is derived and proposed for the quasi-static analysis of large-scale nonlinear structures.

1. Introduction

Structural response under various conditions may be linear or nonliear. Recently, nonlinear analysis of structures has become increasingly common. This is partly due to the use of new design concepts that results in nonlinear structural behavior[1]. Large elastic displacements of long flexible members and local yielding near stress concentrations are typical types of nonlinear behavior. Cable-stayed bridges, guyed towers, and some reticulated structures exhibit significant geometrical nonlinearity under service load. Stress concentrations occurring near abrupt changes in geometry such as a a pressure vessel junction often cause local yielding. Use of new materials often causes overall structural nonlinearity. Safety coniderations may sometimes require that the analysis be carried out into the nonlinear range of behavior even for structures that behave linearly under normal service load.

The cost of a nonlinear analysis is often an order of magnitude greater than that of a linear analysis for the same structure. Nonlinear analyses with the finite element method are generally performed using a series of load increment until the specified total load level is reached. In a load level, a commonly used iterative scheme of the NR(Newton-Raphson) method corrects displacements using a unbalanced force to reach an equilibrium configuration at that load level. The major computational costs arise from repeated formation and factorization of the linearized stiffness matrices. For the economical and efficient computation, the BFGS (Broyden-Fletcher-Goldfarb-Shanno) method or a combined algorithm can be effectively used[2~5].

For large-scale structures, the substructuring scheme can be conveniently used to reduce the input requirements and computation cost by eliminating computation and storage of duplicate information[6~8]. The most significant reduction of computation cost results

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from formation and factorization of substructural equations and condensation. However, most of currently used general-purpose systems such as NASTRAN and ADINA have only a limited substructuring capability for nonlinear analysis[9～11]. Thus, a unified procedure of nonlinear analysis with substructuring has to be developed.

In this paper, the ordinary computational procedure of the BFGS method for the nonlinear finite element solution is reviewed. To check the computational behavior of the BFGS method a small-scale nonlinear structure is analyzed. The NR and BFGS method are used, and the results are compared and discussed. Finally, the promising BFGS algorithm is modified so that it can be used for the nonlinear analysis of large-scale structures with user-controlled substructuring.

2. Characteristics of the BFGS Method

2.1. Basic Procedure

The quasi-static incremental/iterative equilibrium equations for an assemblage of nonlinear finite elements at load level $r+\Delta r$ are[3～5]

$$^{r+\Delta r}K^{(i-1)}\Delta U^{(i)} = {}^{r+\Delta r}P^{(i-1)} \equiv {}^{r+\Delta r}R - {}^{r+\Delta r}Q^{(i-1)} \quad (1)$$

$$^{r+\Delta r}U^{(i)} = {}^{r+\Delta r}U^{(i-1)} + \Delta U^{(i)},\ i=1, 2, \cdots \quad (2)$$

with the initial conditions

$$^{r+\Delta r}U^{(0)} = {}^{r}U,\quad ^{r+\Delta r}Q^{(0)} = {}^{r}Q,\quad ^{r+\Delta r}K^{(0)} = {}^{r}K \quad (3)$$

where an equilibirum at load level r is assumed and,

i=iteration counter

K=tangent stiffness matrix

ΔU=displacement increment

$^{r+\Delta r}R$=external nodal load vector due to body forces, surface loads and concentrated loads at load level $r+\Delta r$ which is usually some fraction of the specified total load

$^{r+\Delta r}Q^{(i-1)}$=nodal point force vector equivalent to the element stresses that correspond to the displacements $^{r+\Delta r}U^{(i-1)}$

$^{r+\Delta r}U^{(i)}$=vector of nodal point displacements at the end of the i-th iteration of the load level $r+\Delta r$

$^{r+\Delta r}P^{(i-1)}$=unbalanced nodal force vector at the end of iteration $(i-1)$.

In the superlinearly convergent NR method, the tangent stiffness matrix $^{r+\Delta r}K^{(i-1)}$ of Eq.(1) is formed using the displacement $^{r+\Delta r}U^{(i-1)}$ in the beginning of each iteration. Consequently, the amount of computation in each iteration of the NR method is significant. To reduce the computational amount in each iteration while maintaining superlinear convergence, the BFGS method can be used. In the method, instead of calculating and factorizing the tangent stiffness, its inverse is directly updated using the formula[2～5]

$$[^{r+\Delta r}K^{(i)}]^{-1} = [A^{(i)}]^{T}[^{r+\Delta r}K^{(i-1)}]^{-1}[A^{(i)}] \quad (4)$$

where T denotes the transpose and $A^{(i)}$ is the updating matrix defined as

$$A^{(i)} = I + V^{(i)}W^{(i)T} \quad (5)$$

In Eq.(5), I=identity matrix, and the vectors $V^{(i)}$ and $W^{(i)}$ are defined as

$$V^{(i)} = -c_i\ {}^{r+\Delta r}P^{(i-1)} - \Delta P^{(i)} \quad (6)$$

$$W^{(i)} = \frac{\Delta U^{(i)}}{\langle \Delta U^{(i)},\ \Delta P^{(i)}\rangle} \quad (7)$$

where the notation of the scalar product $\langle x, y\rangle = x^T y$ has been used, and

$$\Delta P^{(i)} = {}^{r+\Delta r}P^{(i-1)} - {}^{r+\Delta r}P^{(i)} \quad (8)$$

=vector of unbalanced force increment,

$$c_i = \left[\frac{\langle \Delta U^{(i)},\ \Delta P^{(i)}\rangle}{\langle \Delta U^{(i)},\ {}^{r+\Delta r}P^{(i-1)}\rangle}\right]^{1/2} \quad (9)$$

=the condition number of $A^{(i)}$

With the updating formula of Eq.(4), the following requirments for the stiffness updating are satisfied[2]:

(i) quasi-Newton equation; $^{r+\Delta r}K^{(i)}\Delta U^{(i)} = \Delta P^{(i)}$

(ii) hereditary symmetry of the updated stiffness matrix

(iii) hereditary positive definiteness of the stiffness matrix

(iv) inexpensive computation of $\Delta U^{(i)}$

By the requirement of hereditary positive definiteness, a greater guarantee of numerical stability can be obtained. However, an excessively large condition number c_i of Eq.(9) implies that the updated stiffness of Eq.(4) will be nearly singular, which causes numerically vulnerable procedure. To avoid numerically dangerous updates, the updating of Eq·(4) is not performed if the condition number exceeds some preset tolerance c_0, say $c_0=10^6$.

Starting from the initial conditions of Eq.(3), Eq.(4) can be expressed as

$$[{}^{r+\Delta r}K^{(i)}]^{-1}=A^{(i)T}\cdots A^{(1)T}\ [{}^{r}K]^{-1}A^{(1)}\cdots A^{(i)} \qquad (10)$$

Defining the total updating matrix $\bar{A}^{(i)}$ as

$$\bar{A}^{(i)}=A^{(1)}\cdots A^{(i)}=\bar{A}^{(i-1)}A^{(i)},\quad c_i\leq c_0$$
$$\bar{A}^{(i)}=\bar{A}^{(i-1)}\qquad ,\quad c_i>c_0$$
$$\bar{A}^{(0)}=I \qquad (11)$$

an iterative solution of Eq. (1) with numerical stability is given by

$$\Delta U^{(i)}=[\bar{A}^{(i-1)}]^T[{}^{r}K]^{-1}\ [\bar{A}^{(i-1)}]\ {}^{r+\Delta r}P^{(i-1)} \qquad (12)$$

Actual calculation of Eq.(12) involves the following sequence of operations:

(i) Compute an intermediate vector $\bar{P}$ as

$$\bar{P}=\bar{A}^{(i-1)}\ {}^{r+\Delta r}P^{(i-1)} \qquad (13)$$

(ii) Solve for a temporary voctor $\bar{U}$ form

$$^{r}K\bar{U}=\bar{P} \qquad (14)$$

(iii) Obtain $\Delta U^{(i)}$ from

$$\Delta U^{(i)}=\bar{A}^{(i-1)T}\ \bar{U} \qquad (15)$$

To solve Eq.(12), the factorized ^{r}K in Eq.(14) is repeatedly used in each iteration. Thus, significant reduction of computational amount can be achieved.

2.2. Computational Behavior

The three-bar truss of Fig. 1 is taken as an example for studying and numerical evaluating the computational behavior of the BFGS method. The finite element idealization, geometry, and applied load of the truss are also shown in the figure. Two nonlinearity cases(NC) are considered:

(i) NC 1 : geometric nonlinearity only

(ii) NC 2 : both geometric and material nonlinearty

Geometric nonlinearity is modeled by large displacements and small strains through nonlinear strain-displacement relations given as

$$\epsilon_{xx}=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \qquad (16)$$

Nonlinear elastic constitutive equation for material nonlinearity is

$$\sigma_{xx}=E\epsilon_{xx}^{0.5} \qquad (17)$$

For both NC 1 and NC 2, $E=10,000$ ksi is used as material property. As for the cross-sectional area of each member a_i, $i=1,2,3$, and the magnitude of load, two data cases (DC) are used for each NC.

DC 1 : $a_1=8.0$, $a_2=2.0$, $a_3=3.0$(in^2); $R=200$ kips

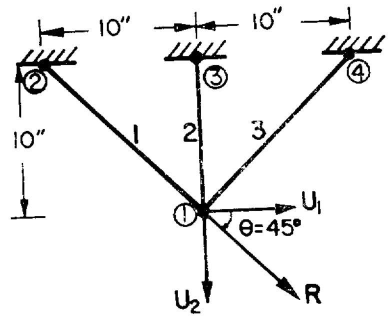

Fig.1 Three-bar truss

DC 2 : $a_1=0.1$, $a_2=0.2$, $a_3=0.2$(in^2); $R=400$ kips

Mild nonlinearity is expected in DC 1 with larger values of cross-sectional area and smaller magnitude of load. Larger nonlinear displacements(more severe nonlinearity) is expected in DC 2 with relatively smaller areas and larger load. For both DCs, we intend to see the computational behavior of the BFGS method.

For the incremental/iterative nonlinear displacement analysis, the final specified loads R are applied in four load steps of $0.25R$, $0.5R$, $0.75R$ and R. In each load level, convergence to an equilibrium is assumed if the norm of unbalanced force vector is less than that of initial unbalanced force with tolerance of 10^{-8}, *i.e*, [3]

$$\|{}^{r+\Delta r}P^{(i)}\|\leq 10^{-8}\|{}^{r+\Delta r}P^{(0)}\| \qquad (18)$$

To Compare the computational behavior of the BFGS method, the NR method is also used. The results from the NR method and the BFGS method are in good agreement at each load level for all the cases. They are shown in Tables 1 and 2 with the results of linear analysis. As shown in Table 1, both the NR and BFGS methods need almost the same number of iterations for each NC where the nonliner displacements are relatively close to the linear ones. However, in DC 2 of Table 2, the difference between the linear and nonlinear displacements becomes larger and the number of iterations is considerably reduced in the BFGS method. This results show that the good convergence property and effective applicability of the BFGS method can be assumed in the structural analysis with higher degree of nonlinearity.

Table 1 Nodal displacements for DC 1

Load level (^rR)	Linear analysis		Nonlinear analysis		Iterations		
			NC1	NC2	method	NC 1	NC 2
1	U_1	0.7367 E-2	0.7364 E-2	0.7228 E-2	NR	3	5
(50)	U_2	0.3792 E-2	0.3794 E-2	0.3897 E-2	BFGS	3	5
2	U_1	0.1473 E-1	0.1472 E-1	0.1435 E-1	NR	3	5
(100)	U_2	0.7584 E-2	0.7593 E-2	0.7876 E-2	BFGS	3	5
3	U_1	0.2210 E-1	0.2207 E-1	0.2140 E-1	NR	3	5
(150)	U_2	0.1138 E-1	0.1140 E-1	0.1190 E-1	BFGS	3	5
4	U_1	0.2947 E-1	0.2941 E-1	0.2840 E-1	NR	3	5
(200)	U_2	0.1517 E-1	0.1521 E-1	0.1597 E-1	BFGS	3	6

Table 2 Nodal displacements for DC 2

Load level (^rR)	Linear analysis		Nonlinear analysis		Iterations		
			NC1	NC2	method	NC 1	NC 2
1	U_1	0.7735 E 0	0.7322 E 0	0.6123 E 0	NR	7	10
(100)	U_2	0.3204 E 0	0.3228 E 0	0.3335 E 0	BFGS	7	8
2	U_1	0.1547 E 1	0.1374 E 1	0.1107 E 1	NR	8	10
(200)	U_2	0.6408 E 0	0.6440 E 0	0.6609 E 0	BFGS	6	7
3	U_1	0.2320 E 1	0.1929 E 1	0.1538 E 1	NR	9	11
(300)	U_2	0.9611 E 0	0.9578 E 0	0.9763 E 0	BFGS	6	7
4	U_1	0.3094 E 1	0.2410 E 1	0.1919 E 1	NR	10	12
(400)	U_2	0.1282 E 1	0.1262 E 1	0.1278 E 1	BFGS	7	7

3. BFGS Method with Substructures

3.1. Substructural Notation

Response of a complex large-scale structural system may be linear or nonlinear. Fortunately, nonlinear behavior usually occurs in isolated regions of most structures. In such cases, a user-defined substructuring, often determined by inspection, enables an engineer to greatly reduce the analysis cost. Even in the cases where the whole structure behaves nonlinearly, substructural formulation can be advantageously used[7].

Using the substructuring concept, a symbolic form of the quasi-static equilibrium equation for finite element model of structural system can be written as

$$\begin{pmatrix} K_{BB} & K_{BI}^{(1)} & \cdot & \cdot & \cdot & K_{BI}^{(s)} \\ K_{IB}^{(1)} & K_{II}^{(1)} & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & K_{II}^{(2)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{IB}^{(s)} & 0 & \cdot & \cdot & \cdot & K_{II}^{(s)} \end{pmatrix} \begin{pmatrix} U_B \\ U_I^{(1)} \\ \cdot \\ \cdot \\ U_I^{(s)} \end{pmatrix} = \begin{pmatrix} R_B \\ R_I^{(1)} \\ \cdot \\ \cdot \\ R_I^{(s)} \end{pmatrix} \tag{19}$$

Here, the symbols K, U, and R may denote stiffness matrix, nodal displacement, and load vector, respectively. The subscripts represent the quantity associated with the substructural boundary (B) and interior(I) degrees of freedom. That is, the subscript B denotes the "master" degree of freedom and I the "slave" degree of freedom. Superscripts represent the substructure number where s is the total number of substructures in the system. For the notational convenience, Eq. (19) is rewritten as

$$\begin{pmatrix} K_{00} & K_{01} & \cdot & \cdot & \cdot & K_{0s} \\ K_{10} & K_{11} & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & K_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{s0} & 0 & \cdot & \cdot & \cdot & K_{ss} \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_s \end{pmatrix} = \begin{pmatrix} R_0 \\ R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_s \end{pmatrix} \tag{20a}$$

or,

$$K_{mn}U_n = R_m \tag{20b}$$

where $m, n=0, 1\cdots s$; and summation convention is assumed for the repeated subscript. Comparing Eqs. (19) and (20), the subscript 0 for m or n means a boundary quantity. Positive integers for m and n denote the interior quantity associated with the m-th and n-th substructure. With this substructural subscript notation, Eqs. (1)～(3) can be written as

$$^{r+\Delta r}K_{mn}^{(i-1)}\Delta U_n^{(i)} = {}^{r+\Delta r}P_m^{(i-1)} \tag{21}$$

$$^{r+\Delta r}U_n^{(i)} = {}^{r+\Delta r}U_n^{(i-1)} + \Delta U_n^{(i)} \tag{22}$$

$$^{r+\Delta r}U_n^{(0)} = {}^{r}U_n, \quad {}^{r+\Delta r}Q_m^{(0)} = {}^{r}Q_m, \quad {}^{r+\Delta r}K_{mn}^{(0)} = {}^{r}K_{mn} \tag{23}$$

Since Eq. (21) is a linearized form of the incremental/iterative substructural equations, it can be easily noted that

$$^{r+\Delta r}K_{mn}^{(i-1)} = {}^{r+\Delta r}K_{nm}^{(i-1)^T}; \text{ for all } m \text{ and } n \tag{24a}$$

$$^{r+\Delta r}K_{mn}^{(i-1)} = 0; \text{ for } \{(m, n) | m\neq 0,\ n\neq 0, \text{ and } m\neq n\} \tag{24b}$$

Using Eq. (24), Eq. (21) can be first solved for the master degrees of freedom and then for the slaves as

$$^{r+\Delta r}\bar{K}_{00}^{(i-1)}\Delta U_0^{(i)} = {}^{r+\Delta r}\bar{P}_0^{(i-1)} \tag{25a}$$

$$\Delta U_n^{(i)} = [{}^{r+\Delta r}K_{nn}^{(i-1)}]^{-1}({}^{r+\Delta r}P_n^{(i-1)} - {}^{r+\Delta r}K_{n0}^{(i-1)}\Delta U_0^{(i)}), \quad n\neq 0 \tag{25b}$$

where the effective master stiffness and the unbalanced load are defined as

$$^{r+\Delta r}\bar{K}_{00}^{(i-1)} = {}^{r+\Delta r}[K_{00} - \sum_{m=1}^{s} K_{0m}K_{mm}^{-1}K_{m0}]^{(i-1)} \tag{26}$$

$$^{r+\Delta r}\bar{P}_0^{(i-1)} = {}^{r+\Delta r}[P_0 - \sum_{m=1}^{s} K_{0m}K_{mm}^{-1}P_m]^{(i-1)} \tag{27}$$

3.2. Substructural BFGS Procedure

In the nonlinear structural analysis, user-defined substructures may be classified as linear and nonlinear ones. A linear (nonlinear) substructure is assumed to behave linearly(nonlinearly) in a specified load level. At each load level, linear and nonlinear substructures may be determined a priori using an appropriate forecasting scheme or by inspection and heuristics. If there exist some linear substructures, the linear degrees of freedom can be statically condensed prior to the iterative solution. Such a scheme has been well-documented [6, 8]. Thus, with no loss of generality, analysis can be performed as if whole structure behave nonlinearly.

Using the substructural notation of Eq. (20), the BFGS updating matrix of Eq. (5) can be written as

$$A_{mn}^{(i)} = \delta_{mn}I + V_m^{(i)}W_n^{(i)^T} \tag{28}$$

where δ_{mn} is the Kronecker's delta and

$$V_m^{(i)} = -c_i{}^{r+\Delta r}P_m^{(i-1)} - \Delta P_m^{(i)} \tag{29}$$

$$W_n^{(i)} = \frac{\Delta U_n^{(i)}}{\sum_{m=0}^{s}\langle \Delta U_m^{(i)},\ \Delta P_m^{(i)}\rangle} \tag{30}$$

The condition number c_i can be calculated as

$$c_i = \left(\frac{\sum_{m=0}^{s}\langle \Delta U_m^{(i)},\ \Delta P_m^{(i)}\rangle}{\sum_{m=0}^{s}\langle \Delta U_n^{(i)},\ {}^{r+\Delta r}P_n^{(i-1)}\rangle} \right)^{1/2} \tag{31}$$

Once the substructural updating matrices of Eq. (28) have been obtained, the substructural forms of total updating matrix are computed as

$$\bar{A}_{mn}^{(i)} = \bar{A}_{mk}^{(i-1)}A_{kn}^{(i)}, \quad \bar{A}_{mn}^{(0)} = \delta_{mn}I \tag{32}$$

Substituting Eq. (28) into Eq. (32),

$$\bar{A}_{mn}^{(i)} = \bar{A}_{mn}^{(i-1)} + \bar{A}_{mk}^{(i-1)}V_k^{(i)}W_n^{(i)^T} \tag{33}$$

With the total updating matrices of Eq. (33) the displacement increments $\Delta U_n^{(i)}$, $n=0, 1, \cdots, s$, can be calculated without much effort. Actual computation sequence of Eqs. (13)～(15) is modified as:

(i) Compute intermediate substructural vectors $\bar{P}_m$, $m=0, 1, \cdots, s$

$$\bar{P}_m = \bar{A}_{mn}^{(i-1)}\,{}^{r+\Delta r}P_n^{(i-1)} \tag{34}$$

(ii) Solve for temporary substructural vectors $\bar{U}_n$, $n=0, 1, \cdots, s$, from

$$^{r}K_{mn}\bar{U}_n = \bar{P}_m \tag{35}$$

At this stage, substructural solution scheme of Eqs.

(25)~(27) can be effectively used. Note that the decomposed forms ${}^{r}K_{mm}, m=1, 2, \cdots, s$, and the effective master stiffness ${}^{r}\bar{K}_{00}$ are repeatedly used in each iteration.

(iii) Calculate $\Delta U_m^{(i)}$, $m=0, 1, \cdots, s$

$$\Delta U_m^{(i)} = \bar{A}_{mn}^{(i-1)^T} \bar{U}_n \quad (36)$$

With the substructural procedure of the BFGS method for the analysis of nonlinear structures, great computational efficiency can be achieved. This is mainly due to the reduction of computational effort and the avoidance of duplicate information. Performing the factorization of small-size substructural matrices instead of a large-scale structural matrix, computational effort can be greatly reduced. Avoiding the treatment of possibly duplicate information, the working space of computer can be also saved.

4. Summary and Conclusion

Substructuring scheme is attractive in the modelling for the analysis of complex large-scale structures. User-defined substructuring is expected to be also advantageous in nonlinear analysis. For the incremental/iterative nonlinear analysis, the BFGS method is shown to be efficient. Thus, a substructural formulation of the BFGS procedure is presented.

The basic BFGS procedure is reviewed and evaluated. Its computational characteristics are also discussed through a small-scale example problem by comparing the results of the NR method. Good convergence property and effective applicability of the BFGS method are again confirmed through a numerical example. Then, the BFGS procedure is modified so that it can be effectively applied in the analysis of large-scale nonlinear structures. Substructuring concepts are used in the modification. Using the modified procedure, computation cost can be reduced. Though the modified substructural BFGS algorithm has not been numerically evaluated yet, it is firmly believed that computational efficiency be also achieved.

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