論 文

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An Adaptive Finite Element Computation for the Added Mass of a Rectangular Cylinder in a Canal

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Abstract

This paper describes an application of the adaptive finite element computations to a free surface flow problem in a canal. A-posteriori error estimates for the adaptive finite element computations are based on the dual extremum principles. Previously the dual extremum principles were applied to compute the upper and lower bounds of the added mass of two-dimensional cylinders in a canal [1, 2]. However, the present method improves the convergence of the computed results by utilizing the local error estimates and by applying the adaptive meshes in the finite element computations. In a test result using triangular elements it is shown that the numerical error in the adaptive finite elements reduces quadratically compared with that in a uniform mesh subdivision.

1. Introduction

The finite element method has been successfully applied to a wide area of engineering problems which were origininally formulated in a form of partial differential equations. While the engineers from many diciplines are investigating further applications to the problems of their interests, some applied mathematicians has made steady efforts towards adaptive mesh refinement through a rigorous local error analysis. The application of adaptive mesh refinement is still in progress and its application is restricted to rather simple mathematical models [3, 4, 5, 6]. It seems to be quite timely that a workshop on this subject was held recently[7].

In this paper an application of the adaptive mesh refinement is described. The basis of the present application is a dual extremum principle for a two-dimensional potential flow problem. A straightforward application of the dual extremum principle has been made to obtain the upper and lower bounds of the added mass for the limiting frequencies in a canal [1]. The theory of the calculus of variations shows that the solutions of certain types of problems is characterized by both a maximum principle and a (different but related) minimum principles, referred to as the dual extremum principles. Theses principles are also known as the complementary variational principles or the upper and lower bounding principles. One of the more well-known pairs of dual extremum principles is the upper- and lower-bound principles associated with the Dirichlet problem of potential theory. We will follow this in the paper. However, one can find a very extensive and systematic derivation of the dual extremum principles in a unified

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account of a diverse range of problems by Noble and Sewell[8].

Now we are interested in estimating the local errors in the computed results. In general the dual extremum principles can not be used for a local error estimation, since the principles state only the maximum and minimum of a quantity which is defined as an integral over the entire domain. However, the dual extremum principle used in Bai [1] can be extended to give a criterion for the local error estimates under an appropriate assumption. Once the local error can be defined, then the application of an adaptive mesh refine ment is straightforward.

The physical problem treated here is an infinitely long rectangular cylinder swaying in an infinitely long canal with a rectangular cross section. This physical problem reduces to a two dimensional problem with a free surface. It is further assumed here that the circular frequency of oscillation becomes zero. Then the classical linearized free surface boundary condition reduces to a rigid wall condition in a potential flow problem.

In this paper the results obtained by an adaptive mesh refinement are compared with those by a uniform mesh refinement. It is shown that the application of the adaptive finite element method is more efficient in the present computations.

2. The Dual Extremum Principles

In this section, we present an application of the dual extremum principles to obtain the upper and lower bounds of the added mass of a two dimensional cylinder in a restricted water. Specifically, the sway added mass for the zero frequency limit is obtained as an application. First, we assume that the x-axis coincides with the free surface and the y-axis is vertically upward. Further we assume that the fluid domain and its boundary are symmetric with respect to the y-axis. Since the sway motion of the rectangular cylinder is assumed, we can treat only half of the fluid domain. In Fig. 1 the boundary configuration is given: the fluid domain is denoted as R, and the boundaries on the body, the free surface, the canal

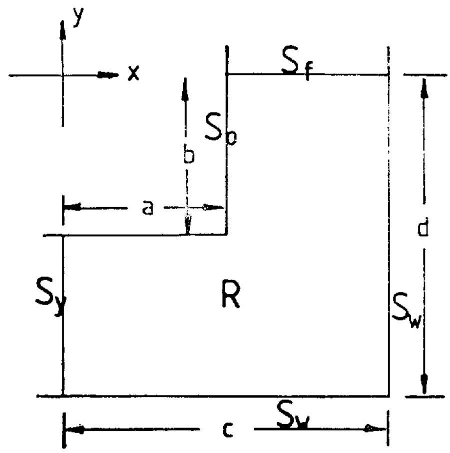

Fig. 1. Boundary Configurations

and the y-axis are denoted by S_0, S_f, S_w, and S_y, respectively. The rectangular body S_0 has the half beam of a and the draft b. The half-breadth and the depth of the canal are c and d, respectively. Hence the mathematical formulation can be written as follows:

$$\begin{aligned} &\nabla^2\phi(x,y)=0 && \text{in } R \\ &\partial\phi/\partial n=n_1 && \text{on } S_0 \\ &\partial\phi/\partial n=0 && \text{on } S_f \text{ and } S_w \\ &\phi=0 && \text{on } S_y \end{aligned} \qquad \text{(1a-d)}$$

where $\vec{n}=(n_1, n_2)$ is outward unit normal vector.

For the goal of contructing a related functional, we formulate the foregoing problem (1) in terms of stream function $\psi(x,y)$ as follows:

$$\begin{aligned} &\nabla^2\psi(x,y)=0 && \text{in } R \\ &\psi=y && \text{on } S_0 \\ &\psi=0 && \text{on } S_f \text{ and } S_w \\ &\partial\psi/\partial n=0 && \text{on } S_y \end{aligned} \qquad \text{(2a-d)}$$

Since the following classical dual-extremum principles are discussed in some detail by Courant and Hilbert [9] and Authurs[10], we give only a brief description here. From the classical theory of calculus of variations, the problem can also be expressed by the following variational principles:

$$\delta J\{\phi\}=0 \qquad \text{(3a)}$$

with the essential condition (1d) where

$$J\{\phi\}=-\iint_R (\nabla\phi)^2 dxdy+2\int_{S0} n_1\phi dS \qquad \text{(3b)}$$

and

$$\delta K\{\psi\}=0 \tag{4a}$$

with the essential conditions (2b) and (2c) where

$$K\{\psi\}=\iint_R (\nabla\psi)^2 dxdy \tag{4b}$$

It is easy to show the equivalence of (1) and (3) or (2) and (4). More details can be found in Bai(1). If ϕ_0 and ψ_0 are the exact solutions of Eqs (1) and (2), respectively, then the velocity fields computed from either the velocity potential or the stream function are the same, i.e.,

$$|\nabla\phi_0(x,y)|=|\nabla\psi_0(x,y)| \text{ at } (x,y)\in R \tag{5}$$

and we have

$$\iint_R (\nabla\phi_0)^2 dxdy=\iint_R (\nabla\psi_0)^2 dxdy. \tag{6}$$

By Green's theorem, (6) reduces to a line integral along the closed boundary of the domain

$$\iint_R (\nabla\phi_0)^2 dxdy=\iint_R (\nabla\psi_0)^2 dxdy=\int_{S0} n_1\phi_0 dS \tag{7}$$

Then the sway added mass for the zero frequency limit, μ, can be defined as

$$\mu=\rho\int_{S0} n_1\phi_0 dS=\rho J_0=\rho K_0 \tag{8}$$

where ρ is the density of the fluid and

$$J_0=J\{\phi_0\}, \qquad K_0=K\{\psi_0\}$$

From this relation, we obtain the following useful inequality:

$$J\{\phi\}\leq\mu/\rho\leq K\{\psi\} \tag{9}$$

The equality holds only when ϕ and ψ are the exact solutions.

3. The Total and Local Error Estimates

The total relative error E_1 in the sway added mass is defined as

$$E_1=\frac{K\{\psi\}-J\{\phi\}}{K\{\psi\}+J\{\phi\}} \tag{10}$$

If we assume that the numerical approximate solutions are close to the exact solutions, ϕ_0 and ψ_0, respectively, then with the relation (5) we can define the following relative local error E_2 as

$$E_2=\frac{1}{\mu_0}\iint_{R_L}[(\nabla\psi)^2-(\nabla\phi)^2]dxdy \tag{11}$$

where R_L is a local domain, i.e., $R_L\subset R$ and μ_0 is the most accurate approximation available to us. Here $\mu_0=2.0728$ of Bai (1) is used.

Further we assume that the errors E_1 and E_2 can be represented, as the number of nodes (N) increases, by

$$E_1=A_1(1/N)^m$$

and

$$E_2=A_2(1/N)^n \tag{12a, b}$$

Here the constants $A_1, A_2, m,$ and n are to be determined from the cumputed results. By taking natural logarithms both sides, Eqs (12a) and (12b) reduce to

$$\log E_1=\log A_1+m\log(1/N) \tag{13a}$$

and

$$\log E_2=\log A_2+n\log(1/N) \tag{13b}$$

4. Numerical Results and Discussions

Computations are made for a rectangular section in a rectangular canal with $a/b=1$, $c/d=1$, $c/a=2$. In the present computation, the half fluid-domin was initially subdivided into 6 equal triangular elements with 8 total nodes as shown in Fig. 2. Two sets of finite element mesh refinements are tested: in the first set, the finite element subdivisions are refined uniformly in the entire domain of computation into a number of equal triangles, while in the second set, the adaptive finite element subdivisions are employed. In the present adaptive mesh subdivision procedures, the element which has the largest local error esti-

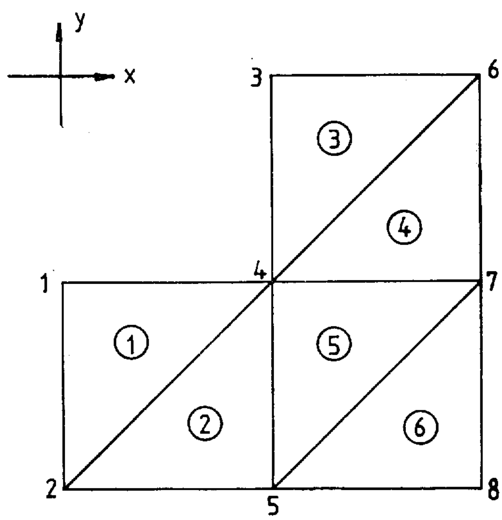

Fig. 2 The initial fiinite-element sudivisions (6 elements and 8 nodes)

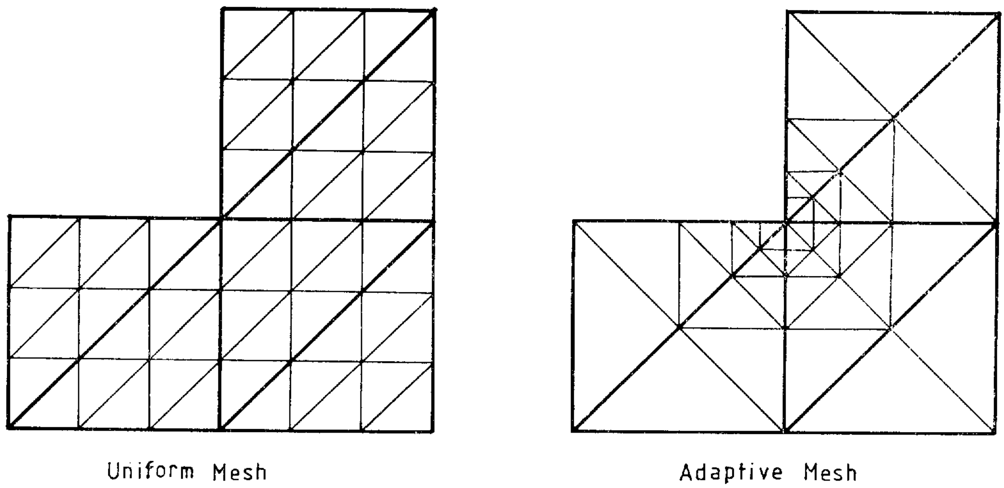

Fig. 3 Finite-element mesh refinements(In both figures, the initial 6 elements are shown by thicker lines)

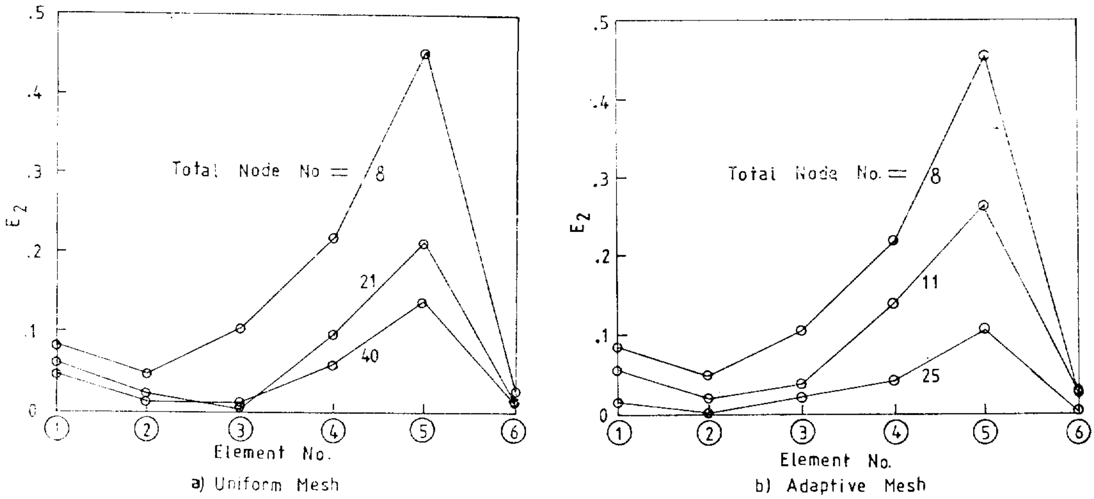

Fig. 4 The convergence of the local errors in two sets of mesh refinements

mate is refined into four equal triangles. During the mesh refinement procedure, the neighboring elements may have to be subdivided in order not to have a unconnected node on their side. This procedure is repeated. Throughout the computations, the local error estimates are summed up for each of the initial 6 elements.

Typical mesh refinements of two different sets are shown in Fig. 3. In Table 1, the total number of elements and nodes in the successive mesh refinements is given. Also given are the upper and lower bounds of the added mass and the maximum local error. The maximum local errors occur always in the 5th element shown in Fig 2. It can be interpreted as the presence of a singularity at the corner gives a strong influence on the present computed results.

Fig. 4 shows the convergence of the local errors in the initial 6 elements. This reuslt shows that the maximum local error reduces faster in the adaptive mesh subdivisions than in the regular uniform mesh subdivisions. It should be noted that it is desirable to obtain a uniform error distribution in all of the 6

Table 1 The uppear and lower bounds of added mass $\mu/2\rho ac$ and maximum local errors for two sets of mesh subdivisions

Set	Node Number	Element Number	Bound		Max. Local Error
			Lower	Upper	
U	8	6	1.50501	3.00000	0.45635
	21	24	1.81282	2.40489	0.21626
	40	54	1.91566	2.26185	0.14461
	65	96	1.96418	2.22007	0.10876
	96	150	1.99171	2.16766	0.08696
A	8	6	1.50501	3.00000	0.45635
	11	12	1.60000	2.50000	0.26534
	15	18	1.76399	2.40513	0.22239
	18	24	1.82324	2.30952	0.16830
	22	30	1.87093	2.26540	0.13525
	25	36	1.89645	2.23478	0.10944
	29	42	1.91335	2.21711	0.09161

U: Uniform mesh subdivisions
A: Adaptive mesh subdivisions

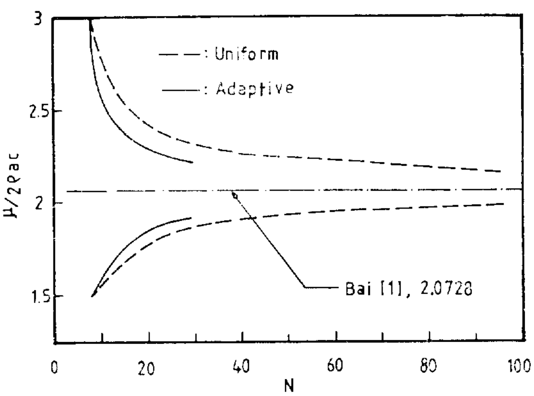

Fig. 5 The upper and lower bounds of the sway added mass

elements. Even though we did not obtain more uniform error distribution here, it suffices to show that the adaptive mesh subdivision provides more uniform local error distributions than the regular uniform mesh subdivision.

Fig. 5 shows the convergence of the upper and lower bounds of the added mass obtained by the two sets of the mesh refinements. The result shows that the accuracy of the added mass computed by the adaptive mesh subdivisions are approximately comparable to that computed by twice as many nodes in

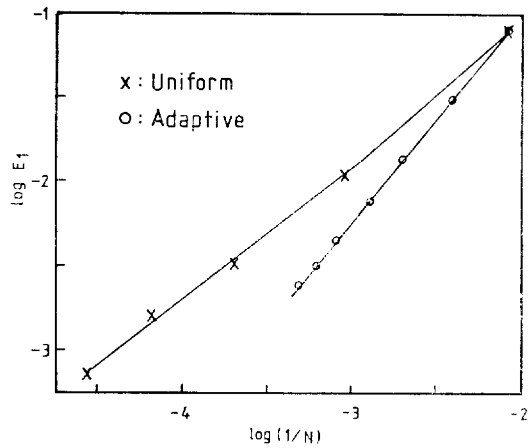

Fig. 6 The convergence test for the error E1

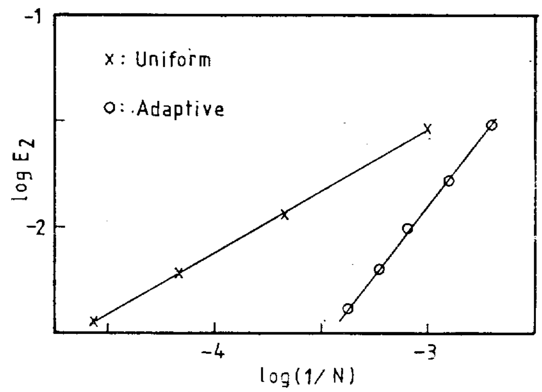

Fig. 7 The convergence test for the maxi mum local error E2

the uniform mesh subdivisions.

Fig 6 and Fig 7 show the convergence of the sway added mass and the maximum local error with respect to the total number of nodes N, respectively The index m defined in Eq (12) is approximately 0.7 in the uniform mesh subdivision and 1.4 in the adptive mesh subdivision. This shows that the rate of convergence is quadratic for the adaptive mesh refinement compared to that for the uniform mesh refinement. The maximum local error shows the similar behabiour: $n=0.6$ and 1.3 for the uniform and adaptive mesh subdivisions, respecitvely.

Acknowledgment

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