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ON THE ω-HYPOELLIPTICITY OF THE BOUNDARY VALUE PROBLEMS

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1. Introduction.

Let $P(D, D_t)$ be a ω-hypoelliptic differential operator of type μ with constant coefficients. Let Ω be an open subset of $\boldsymbol{R}_+^{n+1}$ with open plane piece of boundary σ contained in $\boldsymbol{R}_0^n$. Let $Q_1(D, D_t), \ldots, Q_\mu(D, D_t)$ be μ-partial differential operators with constant coefficients and consider the boundary value problem:

$$(1)\qquad \begin{aligned} &P(D, D_t)u = f \text{ in } \Omega \\ &Q_\nu(D, D_t)u|_\sigma = g_\nu,\ 1 \leq \nu \leq \mu \end{aligned}$$

In this paper we give a necessary condition which may be sufficient, based on the variety of zeros of the characteristic function of the boundary value problem, in order that all C^k-solutions of (1), $k = \max\{\text{order } P, \text{order } Q_\nu,\ 1 \leq \nu \leq \mu\}$, shall belong to $\mathcal{E}_\omega$ whenever the initial data belong to such class (called ω-hypoelliptic boundary value problem).

For completeness we collect basic spaces and results which we need in this paper. Let Ω be an open set in $\boldsymbol{R}^{n+1}$ and ω be a real-valued continuous function on $\boldsymbol{R}^{n+1}$ satisfying the following conditions;

(α) $\qquad 0 = \omega(0) \leq \omega(\xi + \eta) \leq \omega(\xi) + \omega(\eta),\quad \xi, \eta \in \boldsymbol{R}^{n+1}$

(β) $\qquad \displaystyle\int_{\boldsymbol{R}^{n+1}} \frac{\omega(\xi)}{(1+|\xi|)^{n+2}} d\xi < \infty$

(γ) $\qquad \omega(\xi) \geq a + b \log(1+|\xi|)$ for some constants a and $b > 0$

(δ) $\qquad \omega(\xi) \geq \omega(\eta)$ for $|\xi| \geq |\eta|$

In [1] Björck defines $\mathcal{D}_\omega(\Omega)$ the set of all ϕ in $L^1(\boldsymbol{R}^{n+1})$ such that ϕ has compact support in Ω and

$$\|\phi\|_\lambda = \int_{\boldsymbol{R}^{n+1}} |\hat{\phi}(\xi)| e^{\lambda\omega(\xi)} d\xi < \infty$$

for all $\lambda > 0$. Using the inductive limit topology $\mathcal{D}_\omega(\Omega)$ is a Frechet space and elements in $\mathcal{D}_\omega(\Omega)$ are called test functions. The dual space

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of $\mathcal{D}_\omega(\Omega)$ is denote by $\mathcal{D}_\omega'(\Omega)$ whose elements are called generalized distributions on Ω. When $\omega(\xi)=\log(1+|\xi|)$, the generalized distribution space is usual distribution space. $\mathcal{E}_\omega(\Omega)$ is the set of all complex valued functions ψ is Ω such that $\phi\psi$ is in $\mathcal{D}_\omega(\Omega)$ for every ϕ in $\mathcal{D}_\omega(\Omega)$ and the topology is given by the semi-norms $\|\phi\psi\|_\lambda$ for every $\lambda>0$ and every ϕ in $\mathcal{D}_\omega(\Omega)$. The dual space $\mathcal{E}'_\omega(\Omega)$ of the space $\mathcal{E}_\omega(\Omega)$ can be identified with the set of all elements of $\mathcal{D}_\omega'(\Omega)$ which have compact support contained in Ω. The following lemmas can be found in [1].

LEMMA 1. *Let K be a compact convex set in $\boldsymbol{R}^{n+1}$ with support function H. If F is an entire function of $(n+1)$-complex variables $\zeta=\xi+i\eta=(\zeta_1, \ldots, \zeta_{n+1})$, the following three conditions are equivalent:*

(i) *For each $\lambda>0$ and each $\varepsilon>0$ there exists a constant $C_{\lambda,\varepsilon}$ such that for every $\eta\in\boldsymbol{R}^{n+1}$,*

$$\int_{\boldsymbol{R}^{n+1}}|F(\xi+i\eta)|e^{\lambda\omega(\xi)}d\xi\leq C_{\lambda,\varepsilon}e^{H(\eta)+\varepsilon|\eta|}$$

(ii) *For each $\lambda>0$ and each $\varepsilon>0$ there exists a constant $C_{\lambda,\varepsilon}$ such that for $\zeta=\xi+i\eta\in\boldsymbol{C}^{n+1}$.*

$$|F(\xi+i\eta)|\leq C_{\lambda,\varepsilon}e^{H(\eta)+\varepsilon|\eta|-\lambda\omega(\xi)}$$

(iii) $F(\zeta)=\int_{\boldsymbol{R}^{n+1}}e^{-i\langle x,\zeta\rangle}\ \phi(x)dx$ *for some* $\phi\in\mathcal{D}_\omega(K)$.

Using above lemma we can show

LEMMA 2. *Let K be a compact set in $\boldsymbol{R}^{n+1}$, Then the family of semi-norms $\{\phi\rightarrow\|\phi\|_\lambda\}_{\lambda>0}$ on $\mathcal{D}_\omega(K)$ is equivalent to the family of semi-norms*

$$\{\phi\rightarrow\sup_{\zeta\in C^{n+1}}|\hat{\phi}(\zeta)|\exp(\lambda\omega(\xi)-H(\eta)-|\eta|)\}_{\lambda>0}$$

In [3] Hörmander difined the differential operator $P(D)$ is hypoelliptic on Ω when $u\in\mathcal{D}'(\Omega)$ and $P(D)u\in C^\infty(\Omega)$ implies $u\in C^\infty(\Omega)$. He gave a algebraic characterization of these operators as follows;

(2) $\qquad |\mathrm{Im}\ \zeta|\rightarrow\infty$ when $|\zeta|\rightarrow\infty$ on the surface $P(\zeta)=0$.

In [1] Björck extends this concept to ω-hypoelliptic differential operator $P(D)$ in Ω as $u\in\mathcal{D}_\omega'(\Omega)$ and $P(D)u\in\mathcal{E}_\omega(\Omega)$ implies $u\in\mathcal{E}_\omega(\Omega)$ and give an algebraic characterizations of these operators with constant coefficients as follows:

(3) For each $A>0$ there exists B such that $P(\zeta)=0$ implies $|\eta|\geq A\omega(\xi)-B$

In view of both algebraic characterizations and (γ) we can easily see that ω-hypoellipticity implies hypoellipticty.

In all that follows we assume that $P(D, D_t)$ is ω-hypoelliptic differential operator with constant coefficients. We shall consider the root of the equation

(4) $$P(\xi, \tau)=0$$

where $\xi\in\boldsymbol{R}^n$. If τ is a real root, it follows from (2) that ξ belongs to the compact set in $\boldsymbol{R}^n$ defined by $P(\xi, \tau)=0$. That is, if ξ is outside a compact set K (we take a sphere) in $\boldsymbol{R}^n$, (4) has no real roots. Since the roots are continuous function of ξ([2], p239), in each component of the complement of K the number of roots with positive imaginary part is constant. We shall define that $P(D, D_t)$ is of determined type μ if the number of zeros with positive imaginary part is μ for all ξ in the complement of K. When $n>1$ all the ω-hypoelliptic differential operators with constant coefficients are thus of determined type. For ordinary differential operator $k(D_t)$ of order μ, Hörmander shows the following results in [2] which we need in the sequel.

LEMMA 3. *Suppose that all zeros of $k(t)$ have non-negative imaginary parts. Then there is a constant γ depending only on μ such that, if u is a solution of an equation $k(D_t)u=0$, we have*

$$|u(a)|\leq\gamma a^{-1}\int_0^a|u(t)|dt, \quad a>0$$

and

$$\int_0^b|u(t)|dt\leq\left(\frac{a}{b}\right)^\gamma \int_0^a|u(t)|dt, \quad 0<a\leq b.$$

2. ω-hypoelliptic boundary value problems

Let $P(D, D_t)$ be ω-hypoelliptic differential operator of determined type μ with constant coefficients. we shall denote by $\mathcal{A}$ the set of all $\zeta\in\boldsymbol{C}^n$ such that the equation

(5) $$P(\zeta, \tau)=0$$

has exactly μ roots with positive imaginary part and none of that is real. Obviously $\mathcal{A}$ is open in $\boldsymbol{C}^n$ and by hypothesis real ξ is in $\mathcal{A}$ if ξ belongs to a suitable neighborhood of infinity. We shall estimate the size of $\mathcal{A}$ more precisely.

LEMMA 4. *Suppose that $P(D, D_t)$ is ω-hypoelliptic and of determined type μ. Then, given any number $A>0$, there is a constant B such that*

$\mathcal{A}$ contains all ζ satisfying

(6) $$|\mathrm{Im}\ \zeta|<A\omega(\mathrm{Re}\ \zeta)-B.$$

Proof. Taking the same B in (3), we note that if τ is a real and (6) is fulfilled, we have

$$|\mathrm{Im}(\zeta,\tau)|=|\mathrm{Im}\ \zeta|<A\omega(\mathrm{Re}\ \zeta)-B\leq A\omega(\mathrm{Re}\ \zeta,\tau)-B$$

which implies $P(\zeta,\tau)\neq 0$ in virtue of (3). Thus (5) has no real root if (6) is valid and hence the number of roots of (5) with positive imaginary part is constant in each components of the set defined by (6). Now each components of this set contains real points with arbitrarily large absolute values which proves the lemma.

In order to define ω-hypoelliptic boundary value problem on $\Omega\cup\sigma$ we need the following

LAMMA 5. *Let $\phi(x,t)$ be an element in $\mathcal{D}_\omega(\boldsymbol{R}^n\times\boldsymbol{R})$. Then $\phi(x,t)$ is an element in $\mathcal{D}_\omega(\boldsymbol{R}^n)$ for any fixed t in $\boldsymbol{R}$, provided that ω in $\mathcal{D}_\omega(\boldsymbol{R}^n)$ is defined by $\omega(\xi)=\omega(\xi,0)$.*

Proof. Let $\phi_t(x)=\phi(x,t)$ for $x\in\boldsymbol{R}^n$ and $\hat{\phi}_t(\xi)$ the Fourier transform of $\phi_t(x)$ for fixed t, that is,

$$\hat{\phi}_t(\xi)=\int_{\boldsymbol{R}^n}e^{-i<x,\xi>}\phi_t(x)\,dx$$

Using inverse Fourier transform, we have

$$\begin{aligned}\phi(x,t)&=(2\pi)^{-n-1}\int_{\boldsymbol{R}^{n+1}}\hat{\phi}(\xi,\tau)e^{i<x,\xi>}e^{it\tau}d\xi d\tau\\&=(2\pi)^{-n}\int_{\boldsymbol{R}^n}\psi(\xi)e^{i<x,\xi>}d\xi\end{aligned}$$

where $$\psi(\xi)=(2\pi)^{-1}\int_{\boldsymbol{R}}\hat{\phi}(\xi,\tau)e^{it\tau}d\tau=\hat{\phi}_t(\xi)$$

and, for each $\lambda>0$,

$$\begin{aligned}\int_{\boldsymbol{R}^n}|\hat{\phi}_t(\xi)|e^{\lambda\omega(\xi)}d\xi&=\int_{\boldsymbol{R}^n}|\psi(\xi)|e^{\lambda\omega(\xi)}d\xi\\&=\int_{\boldsymbol{R}^n}|(2\pi)^{-1}\int_{\boldsymbol{R}}\hat{\phi}(\xi,\tau)e^{it\tau}d\tau|e^{\lambda\omega(\xi)}d\xi\\&\leq(2\pi)^{-1}\int_{\boldsymbol{R}^{n+1}}|\hat{\phi}(\xi,\tau)|e^{\lambda\omega(\xi)}\ d\xi d\tau\\&\leq(2\pi)^{-1}\int_{\boldsymbol{R}^{n+1}}|\hat{\phi}(\xi,\tau)|e^{\lambda\omega(\xi)}d\xi d\tau\end{aligned}$$

The last term is finite for $\phi\in\mathcal{D}_\omega(\boldsymbol{R}^{n+1})$ and ϕ_t has compact support in $\boldsymbol{R}^n$, which implies $\phi_t\in\mathcal{D}_\omega(\boldsymbol{R}^n)$.

In view of this lemma, the following definition of $\mathcal{E}_\omega(\Omega\cup\sigma)$ make sense.

DEFINITION. $\mathcal{E}_\omega(\Omega\cup\sigma)$ is the set of all u in $\mathcal{E}_\omega(\Omega)$ such that, for any $\phi\in\mathcal{D}_\omega(\sigma)$ and every $j=0, 1, 2, \ldots, \phi(x)D_t^j u(x, t)$ converges in $\mathcal{D}_\omega(\sigma)$ as $t\to 0^+$.

One can easily see that both spaces $\mathcal{E}_\omega(\Omega\cup\sigma)$ and $C^\infty(\Omega\cup\sigma)$ are the same when $\omega(\xi)=\log(1+|\xi|)$. Our definition (1) of ω-hypoelliptic boundary value problem is solution $u\in\mathcal{E}_\omega(\Omega\cup\sigma)$ whenever $f\in\mathcal{E}_\omega(\Omega)$ and $g_\nu\in\mathcal{E}_\omega(\sigma)$, $1\le\nu\le\mu$.

For $\zeta\in A$ we denote by $\tau_1(\zeta), \ldots, \tau_\mu(\zeta)$ the zeros of (5) with positive imaginary part and set

$$k_\zeta(\tau)=\prod_{\nu=1}^{\mu}(\tau-\tau_\nu(\zeta)) \qquad \text{and}$$

$$C(\zeta)=\frac{\det(Q_l(\zeta, \tau_k(\zeta)))1\le k, l\le\mu}{\prod_{l<k}(\tau_k(\zeta)-\tau_l(\zeta))} \tag{7}$$

The function $C(\zeta)$ is called the characteristic function of the boundary value problem (1). $C(\zeta)$ is defined even in the case of repeated roots ([2], p231).

Our main result is to prove the following theorem

THEOREM 6. *Suppose the boundary value problem* (1) *is ω-hypoelliptic on $\Omega\cup\sigma$. Then, for every number $A>0$ there is a constant B such that*

$$\zeta\in C^n \text{ and } |\text{Im } \zeta|<A\omega(\text{Re } \zeta)-B \tag{8}$$
$$\text{implies } \zeta\in A \text{ and } C(\zeta)\neq 0$$

Proof. In order to avoid unnecessarily complications we assume that Ω is bounded. Let H $(\Omega\cup\sigma)$ be the set of all $u\in C^k(\Omega\cup\sigma)$, which satisfy the homogeneous boundary value problem

$$P(D, D_t)u=0 \qquad \text{in } \Omega \tag{9}$$
$$Q_\nu(D, D_t)u|_\sigma=0, \ 1\le\nu\le\mu$$

and have finite norm

$$N(u)=\sup_{\substack{x\in\Omega\\|\alpha|\le k}} |D^\alpha u(x)|$$

Then H $(\Omega\cup\sigma)$ is a Banach space with respect to this norm. Take an finite open ball σ' in σ and Ω' an open cylinder with height a in Ω. Consider the following semi-norms on $\mathcal{E}_\omega(\Omega'\cup\sigma')$; For each positive real $\lambda, \phi\in\mathcal{D}_\omega(\Omega')$ and $\psi\in\mathcal{D}_\omega(\sigma')$, define

$$P^{\lambda}_{\phi,\psi}(u)=\|\phi u\|_{\lambda}+\sup_{\substack{|\alpha|\leq k\\ 0<t<a}}\|\psi(D^{\alpha}u)_t\|_{\lambda}$$

for all $u\in\mathcal{E}_{\omega}(\Omega'\cup\sigma')$. According to definition of $\mathcal{E}_{\omega}(\Omega\cup\sigma)$ it is easily seen that $P^{\lambda}_{\phi,\psi}$ are semi-norms on $\mathcal{E}_{\omega}(\Omega'\cup\sigma')$ and with these semi-norms $\mathcal{E}_{\omega}(\Omega'\cup\sigma')$ is a Frechet space. By the hypoellipticity of the boundary value problem (9) and the closed graph theorem the restriction mapping from $H(\Omega\cup\sigma)$ to $\mathcal{E}_{\omega}(\Omega'\cup\sigma')$ is continuous.

Fix $\psi\in\mathcal{D}_{\omega}(\sigma')$ with $\hat{\psi}(0)\neq 0$. For each real $\lambda>0$ and each $\phi\in\mathcal{D}_{\omega}(\Omega')$ there is a constant $C_{\lambda,\phi}$, from the continuity of the restriction map, such that

(10) $$P^{\lambda}_{\phi,\psi}(u)\leq C_{\lambda,\phi}\sup_{\substack{|\alpha|\leq k\\ (x,t)\in\Omega}}|D^{\alpha}u(x,t)|$$

for every $u\in H(\Omega\cup\sigma)$.

Now suppose $\zeta_0\in A$ and $C(\zeta_0)=0$. Then there is a non-trivial solution $v(t)$ (see [2]) of the equetion

(11) $$k_{\zeta_0}(D_t)v(t)=0$$
$$(Q_{\nu}(\zeta_0,D_t)v)(0)=0,\quad 1\leq\nu\leq\mu.$$

The solution $u(x,t)=e^{i\langle x,\zeta_0\rangle}v(t)$ of (9) is contained in $H(\Omega\cup\sigma)$. Substituting this solution in (10), we have

(12) $$\sup_{\substack{|\alpha|\leq k\\ 0<t<a}}\|\psi(D^{\alpha}u)_t\|_{\lambda}\leq C_{\lambda}\sup_{\substack{|\alpha|\leq k\\ (x,t)\in\Omega}}|D^{\alpha}u(x,t)|$$

Since, from lemma 2, on supp (ψ) the semi-norms $\|\widehat{\psi(D^{\alpha}u)_t}\|_{\lambda}$ are equivalent to the semi-norms

$$\sup_{\zeta\in C^n}|\widehat{\psi(D^{\alpha}u)}_t(\zeta)|e^{\lambda\omega(\xi)-H(\eta)-|\eta|}$$

with the support function H of supp (ψ), (12) can be written

(13) $$\sup_{\substack{|\alpha|\leq k\\ 0<t<a}}\sup_{\zeta\in C^n}|\widehat{\psi(D^{\alpha}u)}_t(\zeta)|e^{\lambda\omega(\xi)-H(\eta)-|\eta|}$$
$$\leq C_{\lambda}\sup_{\substack{|\alpha|\leq k\\ (x,t)\in\Omega}}|D^{\alpha}u(x,t)|$$

Using the facts that $\widehat{\psi(D^{\alpha}u)}_t(\zeta)=v_{\alpha}(t)\hat{\psi}(\zeta-\zeta_0)$ with $v_{\alpha}(t)e^{i\langle x,\zeta_0\rangle}=D^{\alpha}(e^{i\langle x,\zeta_0\rangle}v(t))$ and $|D^{\alpha}u(x,t)|=e^{-\langle x,\mathrm{Im}\zeta_0\rangle}|v_{\alpha}(t)|$, (13) will be

(14) $$\sup_{\substack{|\alpha|\leq k\\ 0<t<a}}\left(\sup_{\zeta\in C^n}|\hat{\psi}(\zeta-\zeta_0)|e^{\lambda\omega(\xi)-H(\eta)-|\eta|}\right)|v_{\alpha}(t)|$$
$$\leq C_{\lambda}\sup_{\substack{|\alpha|\leq k\\ (x,t)\in\Omega}}e^{-\langle x,\mathrm{Im}\zeta_0\rangle}|v_{\alpha}(t)|.$$

Substituting ζ_0 into ζ in (14), we have

(15) $$\sup_{\substack{|\alpha|\leq k \\ 0<t<a}} |\hat{\phi}(0)|e^{\lambda\omega(\xi_0)-H(\eta_0)-|\eta_0|}|v_\alpha(t)|$$
$$\leq C_\lambda e^{H_1(\eta_0)} \sup_{\substack{|\alpha|\leq k \\ (x,t)\in\Omega}} |v_\alpha(t)|$$

with the support function H_1 of Ω. Applying lemma 3 with $b=\sup_{(x,t)\in\Omega} t$ which is greater than a for solutions $v_\alpha(t)$ of $k_{\zeta_0}(D_t)$ we obtain

(16) $$\hat{\phi}(0)\ e^{\lambda\omega(\xi_0)-H(\eta_0)-|\eta_0|} \sup_{\substack{|\alpha|\leq k \\ 0<t<a}} |v_\alpha(t)|$$
$$\leq C_\lambda \gamma\left(\frac{b}{a}\right)^{r-1} (\sup_{\substack{|\alpha|\leq k \\ 0<t<a}} |v_\alpha(t)|)e^{H_1(\eta_0)}$$

for a constant γ depending only on μ. Since $v(t)$ is a non-trivial solution of (11), we have the following inequality from (16) with different constant C_λ:

$$e^{\lambda\omega(\xi_0)-H(\eta_0)-|\eta_0|}\leq C_\lambda e^{H_1(\eta_0)}$$

which implies

(17) $$H(\eta_0)+H_1(\eta_0)+|\eta_0|\leq\lambda\omega(\xi_0)-\log C_\lambda$$

Since λ is arbitrary and k does not depend on λ, for given $A>0$ we can take λ with $\lambda>A(k+1)$ and suitable constant $B>\frac{1}{k+1}$ log C_λ, so that

(19) $$|\eta_0|\geq A\omega(\xi_0)-B$$

for any $\zeta_0\in\mathcal{A}$ and $C(\zeta_0)=0$. This implies our result.

REMARK. We showed that, in [4] and [5] the condition (8) for $\omega(\xi)=\log\ (1+|\xi|)$ and $\omega(\xi)=|\xi|^{\frac{1}{d}}$ is a characterization of the hypoellipticity and Gevrey hypoellipticity for the boundary value problem, respectively. Therefore we expect that the condition (8) for general ω shall be sufficient condition of ω-hypoellipticity.

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