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FUNCTIONS OF $\kappa\Phi$-BOUNDED VARIATION

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In defining a function of bounded variation on the closed interval $[a, b]$ we considered the supremum of $\sum|f(I_n)|$ for every collection $\{I_n\}$ of nonoverlapping subintervals of $[a, b]$ such that $[a, b]=\bigcup I_n$ where $f(I_n)=f(y_n)-f(x_n)$, $I_n=[x_n, y_n]$. A function f is of bounded variation on $[a, b]$ if $V_a^b(f)=\sup\sum|f(I_n)|<\infty$. Equivalently we could say a function is of bounded variation on the closed interval $[a, b]$ if there exists a positive constant C such that for every collection $\{I_n\}$ of subintervals of $[a, b]$, $\sum|f(I_n)|\leqq C$. Daniel S. Cyphert [1] generalized this idea by considering other functions κ on $[0, 1]$ in his Ph. D dissertation. f is said to be of κ-bounded variation on $[a, b]$ if there exists a positive constant C such that for every collection $\{I_n\}$ of nonoverlapping subintervals of $[a, b]$, $\sum|f(I_n)|\leqq C\sum\kappa\left(\frac{|I_n|}{b-a}\right)$ where $|I_n|=y_n-x_n$, $I_n=[x_n, y_n]$. On the other hand, Michael Schramm [2] generalized the above idea by considering a sequence of increasing convex functions $\Phi=\{\phi_n\}$ defined on $[0, \infty]$: f is of Φ-bounded variation on $[a, b]$ if $V_\Phi(f;a, b)=\sup\sum\phi_n(|f(I_n)|)<\infty$ where the supremum being taken over all nonoverlapping subintervals $\{I_n\}$, $I_n\subset[a, b]$. We are going to combine the above two concepts.

The introduction of the function κ can be viewed as a rescaling of lengths of intervals in $[a, b]$ such that the length of $[a, b]$ is 1 if $\kappa(1)=1$. Indeed we now require throughout the following that κ has the following properties on $[0, 1]$:

(1) κ is continuous with $\kappa(0)=0$ and $\kappa(1)=1$.

(2) κ is concave and strictly increasing, and

(3) $\lim_{x\to0^+}\frac{\kappa(x)}{x}=\infty$.

Note: If $\lim_{x\to0^+}\frac{\kappa(x)}{x}=c$, the set κBV of κ-bounded variation functions

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is the set BV of bounded variation functions. So, to enlarge the class of functions under consideration we have imposed the condition (3).

EXAMPLE. $\kappa_0(x)=\begin{cases} x(1-\log x), & x\neq 0 \\ 0 & , x=0 \end{cases}$

$\kappa_\alpha(x)=x^\alpha$, for $0<\alpha<1$.

Let $\Phi=\{\phi_n\}$ be a sequence of increasing convex functions defined on nonnegative numbers and such that $\phi_n(0)=0$, $\phi_n(x)>0$.

DEFINITION. Let a real valued function f be defined on the closed interval $[a, b]$. f is said to be of $\kappa\Phi$-bounded variation on $[a, b]$ if there exists a positive constant C such that for every collection $\{I_n\}$ of non-overlapping subintervals of $[a, b]$ such that $[a, b]=\cup I_n$

$$\sum\phi_n(|f(I_n)|)\leq C\sum\kappa\left(\frac{|I_n|}{b-a}\right),$$

where $|I_n|$ is the length of I_n. The total $\kappa\Phi$-variation of f over $[a, b]$ is defined by $\kappa V_\Phi(f)=\kappa V_\Phi(f;\ a, b)=\sup\dfrac{\sum\phi_n(|f(I_n)|)}{\sum\kappa\left(\frac{|I_n|}{b-a}\right)}$, where the supremum is taken over all nonoverlapping subintervals $\{I_n\}$, $I_n\subset[a, b]$.

We denote by $\kappa\Phi BV$ the collection of all $\kappa\Phi$-bounded variation functions on $[a, b]$. If we take $\phi_n(x)=x$ for all n, then $\kappa\Phi BV=\kappa BV$ examined in [1]. If we take $\kappa(x)=x$, then $\kappa\Phi BV=\Phi BV$ examined in [2].

By the above definitions, we have the following

THEOREM 1. 1) *For fixed* $\Phi=\{\phi_n\}$ *and* $\kappa_1\leqq\kappa_2$, *we have* $\kappa_1\Phi BV\subset\kappa_2\Phi BV$

2) *For fixed* κ *and* $\Phi_1=\{\phi_{1n}\}$, $\Phi_2=\{\phi_{2n}\}$, $\phi_{1n}\geqq\phi_{2n}$, *we have* $\kappa\Phi_1 BV\subset\kappa\Phi_2 BV$

3) *For* $\kappa_1\leq\kappa_2$ *and* $\Phi_1=\{\phi_{1n}\}$, $\Phi_2=\{\phi_{2n}\}$, $\phi_{1n}\geqq\phi_{2n}$, *we have* $\kappa_1\Phi_1 BV\subset\kappa_2\Phi_2 BV$.

If $L(x)$ is defined for $x\geq 0$, $L(0)=0$ and L is concave then L is subadditive, i.e., $L(a+b)\leqq L(a)+L(b)$, since

$$\frac{L(a+b)-L(b)}{(a+b)-b}\leqq\frac{L(a)-L(0)}{a-0}.$$

THEOREM 2. *Suppose that* f *is of* Φ*-bounded variation on a closed interval* $[a, b]$. *Then* f *is* $\kappa\Phi$*-bounded variation on the closed interval* $[a, b]$ *and* $\kappa V_\Phi(f)\leqq V_\Phi(f)$ *i.e.,* $\Phi BV\subset\kappa\Phi BV$.

Proof. For every collection $\{I_n\}$ of nonoverlapping subintervals of $[a, b]$ such that $[a, b] = \cup I_n$,

$$\begin{aligned}\sum\phi_n(|f(I_n)|) &\le V_\Phi(f) \\ &= V_\Phi(f)\kappa\left(\sum\frac{|I_n|}{b-a}\right) \\ &= V_\Phi(f)\sum\kappa\left(\frac{|I_n|}{b-a}\right)\end{aligned}$$

since $1=\kappa(1)=\kappa\left(\sum\frac{|I_n|}{b-a}\right)\le\sum\kappa\left(\frac{|I_n|}{b-a}\right)$ by subadditivity of κ.

Note: $BV\subsetneqq\kappa BV$ in [1].

Let us consider $\kappa V_\Phi(cf)$ as a function of variable c. If $\Phi=\{\phi_n\}$ is a sequence of increasing convex functions, $\phi_n(0)=0$, $x\ge 0$, we have $\phi_n(cx)\le c\phi_n(x)$, $0\le c\le 1$. Let $\kappa V_\Phi(f)<\infty$ and let $0<c\leqq 1$. Then $\kappa V_\Phi(cf)\leqq c\kappa V_\Phi(f)\to 0$ as $c\to 0$. With this in mind, we define a norm as follows:

Let $\kappa\Phi V_0=\{f\in\kappa\Phi BV \mid f(a)=0\}$. For $f\in\kappa\Phi BV_0$, let $|||f|||=|||f|||_{\kappa\Phi}=\inf\{k>0 \mid \kappa V_\Phi(f/k)\leqq 1\}$.

LEMMA 3. 1) $\kappa V_\Phi(f/|||f|||)\leqq 1$

2) *If* $|||f|||\leqq 1$, *then* $\kappa V_\Phi(f)\le |||f|||$.

Proof. 1) Take $k>|||f|||$, then for any collection $\{I_n\}$

$$\frac{\sum\phi_n(|f(I_n)|/k)}{\sum\kappa(|I_n|/b-a)}\leqq\kappa V_\phi(f/k)\leqq 1.$$

Thus

$$\kappa V_\Phi(f/|||f|||)=\sup\lim_{k\to|||f|||+}\frac{\sum\phi_n(|f(I_n)|/k)}{\sum\kappa\left(\frac{|I_n|}{b-a}\right)}\le 1.$$

2) For any $\{I_n\}$, since $|||f|||\leqq 1$,

$$\sum\phi_n(|f(I_n)|)/\sum\kappa(|I_n|/b-a)\leqq |||f|||\left(\frac{\sum\phi_n(|f(I_n)|/|||f|||)}{\sum\kappa(|I_n|/b-a)}\right)\le |||f|||.$$

By using this lemma, we have the following result with the similar proof for ΦBV_0 in [2].

THEOREM. $(\kappa\Phi BV_0, |||\cdot|||)$ *is a Banach space.*

$\kappa\Phi BV$ may be made a Banach space with norm $|f(a)|+|||f-f(a)|||$.

DEFINITION. Let a function f be defined on the interval $[a, b]$. f is said to be $\kappa\Phi$-decreasing on $[a, b]$ if there exists a positive constant C such that

$$\phi_n(|f(I)|)\leq C\kappa\left(\frac{|I|}{b-a}\right)$$

for any interval $I\subset[a,b]$.

Just as every decreasing function is of bounded variation and every κ-decreasing function is of κ-bounded variation, we have the following obvious generalization of $\kappa\Phi$-decreasing function.

LEMMA 4. *Suppose that a function f is $\kappa\Phi$-decreasing on $[a,b]$. Then f is of $\kappa\Phi$-bounded variation.*

Although κ-decreasing functions need not be continuous, $\kappa\Phi$-decreasing functions are always continuous.

LEMMA 5. *Suppose that f is $\kappa\Phi$-decreasing on the closed interval $[a,b]$. Then for each $a\leq x_0<b$, and $a<y_0\leq b$, $f(x_0^+)$ and $f(y_0^-)$ exist.*

Proof. Put $B=\overline{\lim}_{x\to x_0^+} f(x)\geq \underline{\lim}_{x\to x_0^+} f(x)=A$. Let $\{x_i\}_{i=1}^{\infty}$ and $\{x_i'\}_{i=1}^{\infty}$ be the sequences converging to x_0 such that $\lim_{n\to\infty} f(x_n)=B$ and $\lim_{n\to\infty} f(x_n')=A$. We may assume without loss of generality that $x_1>x_1'>x_2>x_2'>\cdots$. So

$$\phi_1(|f(x_n)-f(x_n')|)\leq c\kappa\left(\frac{x_n-x_n'}{b-a}\right).$$

Taking limit on both sides gives $\phi_1(|B-A|)\leq 0$ and therefore $B-A=0$ and f has a right hand limit at x_0. A similar argument shows that $f(y_0^-)$ exists.

Moreover we have the following

THEOREM 6. *Suppose that f is $\kappa\Phi$-decreasing on the closed interval $[a,b]$. Then f is continuous on $[a,b]$.*

Proof. Let $a\leq x<x_0<y\leq b$. Then $\phi_1(|f(x)-f(x_0)|)\leqq C\kappa\left(\frac{x_0-x}{b-a}\right)$ and $\phi_1(|f(y)-f(x_0)|)\leqq C\kappa\left(\frac{y-x_0}{b-a}\right)$. Letting $x\to x_0$ and $y\to x_0$ shows $\phi_1(|f(x^+)-f(x_0)|)\leq 0$ and $\phi_1(|f(x_-)-f(x_0)|)\leqq 0$. The existence of $f(x^+)$ and $f(x_-)$ follows by lemma 5. Thus $f(x^+)=f(x_0)=f(x_-)$.

THEOREM 7. *Suppose that $\Phi_1=\{\phi_{1n}\}$, $\Phi_2=\{\phi_{2n}\}$ and $\Phi_3=\{\phi_{3n}\}$ satisfy $\phi_{1n}^{-1}(x)\phi_{2n}^{-1}(x)\leqq\phi_{3n}^{-1}(x)$ for all n. Then $fg\in\kappa\Phi_3BV$ for all $f\in$*

$\kappa\Phi_1 BV$, $g \in \kappa\Phi_2 BV$ *and*

$$\|fg\|_{\kappa\Phi_3} \leqq 2 \|f\|_{\kappa\Phi_1} \|g\|_{\kappa\Phi_2} \text{ for } f(a) = g(a) = 0.$$

Proof. Given any $I_n \subset [a, b]$, either $\phi_{1n}(|f(I_n)|) \leqq \phi_{2n}(|g(I_n)|)$ or $\phi_{1n}(|f(I_n)|) > \phi_{2n}(|g(I_n)|)$. If $\phi_{1n}(|f(I_n)|) \leqq \phi_{2n}(|g(I_n)|)$ then

$$\begin{aligned} |f(I_n)g(I_n)| &= \phi_{1n}^{-1}(\phi_{1n}(|f(I_n)|))\phi_{2n}^{-1}(\phi_{2n}(|g(I_n)|)) \\ &\leqq \phi_{1n}^{-1}(|\phi_{2n}(|g(I_n)|))\phi_{2n}^{-1}(\phi_{2n}(|g(I_n)|)) \\ &\leqq \phi_{3n}^{-1}(\phi_{2n}(|g(I_n)|)). \end{aligned}$$

Thus $\phi_{3n}(|f(I_n)g(I_n)|) \leqq \phi_{2n}(|g(I_n)|)$.
If $\phi_{1n}(|f(I_n)|) > \phi_{2n}(|g(I_n)|)$, a similar argument shows that $\phi_{3n}(|f(I_n)g(I_n)|) \leq \phi_{1n}(|f(I_n)|)$.
Therefore

$$\frac{\sum \phi_{3n}(|f(I_n)g(I_n)|)}{\sum \kappa\left(\frac{|I_n|}{b-a}\right)} \leqq \frac{\sum \phi_{1n}(|f(I_n)|)}{\sum \kappa\left(\frac{|I_n|}{b-a}\right)} + \frac{\sum \phi_{2n}(|g(I_n)|)}{\sum \kappa\left(\frac{|I_n|}{b-a}\right)}$$

Thus $fg \in \kappa\Phi_3 BV$.

Let $\varepsilon > 0$, without loss of generality assume $\|f\|_{\Phi_1} = 1 = \|g\|_{\Phi_2}$. By the convexity of $\phi_{3n}(x)$.

$$\frac{\sum \phi_{3n}(|f(I_n)g(I_n)|/2(1+\varepsilon)^2)}{\sum \kappa\left(\frac{|I_n|}{b-a}\right)} \leqq \frac{\sum \frac{1}{2} \phi_{3n}\left(\frac{|f(I_n)|}{1+\varepsilon} \frac{|g(I_n)|}{1+\varepsilon}\right)}{\sum \kappa\left(\frac{|I_n|}{b-a}\right)}$$

$$\leqq \frac{\frac{1}{2} \sum \phi_{1n}\left(\frac{|f(I_n)|}{1+\varepsilon}\right)}{\sum \kappa\left(\frac{|I_n|}{b-a}\right)} + \frac{\frac{1}{2} \sum \phi_{2n}\left(\frac{|g(I_n)|}{1+\varepsilon}\right)}{\sum \kappa\left(\frac{|I_n|}{b-a}\right)} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Thus $\kappa V_{\Phi_3}(fg/2(1+\varepsilon)^2) \leqq 1$, $\|fg\|_{\kappa\Phi_3} \leq 2(1+\varepsilon)^2$ and the theorem follows by letting $\varepsilon \to 0$.

References

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