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A NOTE ON THE OPERATOR EQUATION $\alpha+\alpha^{-1}=\beta+\beta^{-1}$

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1. Introduction

Let M be a von Neumann algebra and α, β be $*$-automorphisms of M satisfying the operator equation

$$\alpha+\alpha^{-1}=\beta+\beta^{-1}$$

This operator equation has been extensively studied and many important decomposition theorems have been obtained by several authors (for instance see [4], [5], [2], [1]). Originally, this operator equation arose in the paper of Van Daele on the new approach of the Tomita-Takesaki theory in the case of modular operators ([7]). In the case of one-parameter automorphism groups, this equation has produced a bounded and completely positive map which can play a role similar to the infinitesimal generator (for details see [6] and [1]). A recent and one of the most important applications of this equation has been in developing an analogue of the Tomita-Takesaki theory for Jordan algebras by Haagerup [3]. One general result of this theory is the following

THEOREM. *Let M be a Von Neumann algebra and α, β be commuting $*$-automorphisms of M satisfying the operator equation $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. Then there exists a central projection p in M such that $\alpha(px)=\beta(px)$ and $\alpha((1-p)x)=\beta^{-1}((1-p)x)$ for all x in M (see [4], [5] and [2]). The proof of this theorem depends, in one or the other form, on the deep algebraic and topological properties of von Neumann algebras.*

Important applications of the above theorem are reflected in the case when the von Neumann algebra is $B(H)$ (the algebra of all bounded operators on a Hilbert space H) or a factor in which case this operator equation has a complete solution in the sense that either $\alpha=\beta$ or $\alpha=\beta^{-1}$ which can be obtained from the above theorem by putting $p=1$ (see, for instance, [4] and [5]). In view of the importance of this

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conclusion for $B(H)$ or factors, it is worthwhile to look for an independent and simpler proof of this result in this case. We give here a direct and simpler proof of this result without using the technical properties of von Neumann algebras and thus is easily comprehensible to a reader with elementary background in operator theory.

2. Results

We prove the following result.

THEOREM 2.1. *Let α, β be $*$-automorphisms on $B(H)$ such that*

$$\alpha+\alpha^{-1}=\beta+\beta^{-1}$$

then either $\alpha=\beta$ or $\alpha=\beta^{-1}$.

Proof. It is well-known that α, β are inner, so there exist unitaries u and v such that

$$\alpha(x)=uxu^*$$
$$\beta(x)=vxv^*$$

for all x in $B(H)$.
Thus, we have

$$uxu^*+u^*xu=vxv^*+v^*xv. \tag{1}$$

We can rewrite equation (1) in terms of Hilbert-Schmidt operators on $H\otimes\bar{H}$ as:

$$u\otimes\bar{u}+u^*\otimes\bar{u}^*=v\otimes\bar{v}+v\otimes\bar{v}^* \tag{2}$$

Assume that $u^*\neq\lambda u$ for any complex number λ (and similarly for v). Let W be in $B(H)_*$ such that $W(v)=1$ and $W(v^*)=0$. Applying $(1\otimes\bar{W})$, we get

$$\bar{W}(\bar{u})u+\bar{W}(\bar{u}^*)u^*=v.$$

So there exist numbers k_1 and k_2 such that

$$v=k_1u+k_2u^*$$
$$v^*=\bar{k}_1u^*+\bar{k}_2u$$

and hence from equation (2), we have

$$\begin{aligned}u\otimes\bar{u}+u^*\otimes\bar{u}^*=&\{|k_1|^2+|k_2|^2\}u\otimes\bar{u}+(k_1\bar{k}_2+\bar{k}_2k_1)u\otimes\bar{u}^*\\&+(\bar{k}_1k_2+k_2\bar{k}_1)u^*\otimes\bar{u}+(|k_1|^2+|k_2|^2)u^*\otimes\bar{u}^*.\end{aligned}$$

Since u and u^* are linearly independent, it follows that $k_1\bar{k}_2=0$ and

$|k_1|^2+|k_2|^2=1$. Now $k_1\bar{k}_2=0$ implies either $k_1=0$ or $k_2=0$. So, if $k_1=0$ then $|k_2|=1$ and if $k_2=0$ then $|k_1|=1$. In the first case $v=k_2u^*$ and hence

$$\beta^{-1}(x)=v^*xv=\bar{k}_2uxk_2u^*=|k_2^2|uxu^*=\alpha(x)$$

for all x in $B(H)$.

In the second case, $v=k_1u$ and by similar calculations, we have that

$$\alpha(x)=\beta(x) \text{ for all } x \text{ in } B(H).$$

Thus we have either $\alpha(x)=\beta(x)$ for all x in $B(H)$ or $\alpha(x)=\beta^{-1}(x)$ for all x in $B(H)$.

Similarly, when $u^*=\lambda u$ for a complex number λ, with $|\lambda|=1$, we get

$$2(u\otimes\bar{u})=v\otimes\bar{v}+v^*\otimes\bar{v}^*.$$

Again choosing W in $B(H)_*$ with $W(v)=1$ and $W(v^*)=0$, we have

$$2\bar{W}(\bar{u})u=v.$$

This is possible only when $|2\bar{W}(\bar{u})|=1$ and this implies that $\alpha=\beta$.

So in any case, either $\alpha=\beta$ or $\alpha=\beta^{-1}$. Q. E. D.

The following corollary is, in fact, an improvement of Theorem 2. 1.

COROLLARY 2. 2. *Let α, β be inner $*$-automorphisms of a factor M acting on a Hilbert space H such that*

$$\alpha+\alpha^{-1}=\beta+\beta^{-1}$$

then, either $\alpha=\beta$ or $\alpha=\beta^{-1}$.

Proof. Let u and v be unitaries such that

$$\alpha(x)=uxu^*$$
$$\text{and} \quad \beta(x)=vxv^*$$

for all $x\in M$.

We define $\tilde{\alpha}$ and $\tilde{\beta}$ on $B(H)$ by the same formulas.

Because M is a factor and $(M\cup M')'=M\cap M'=\mathbf{C}1$, we get that $(M\cup M')''=B(H)$. Therefore, the algebra generated by M and M' is $B(H)$ and hence for any $x\in B(H)$, we can write

$$x=\sum_{i=1}^{n}a_ia_i'$$

where $a_i\in M$ and $a_i'\in M'$.

Remark that $\tilde{\alpha}(a_i')=a_i'$ and $\tilde{\alpha}^{-1}(a_i')=a_i'$ (because $u\in M$).

Applying $\tilde{\alpha}+\tilde{\alpha}^{-1}$ on $B(H)$, we get

$$(\tilde{\alpha}+\tilde{\alpha}^{-1})(x)=\sum_{i=1}^{n}\alpha(a_i)a_i'+\sum_{i=1}^{n}\alpha^{-1}(a_i)a_i'$$
$$=\sum_{i=1}^{n}(\alpha+\alpha^{-1})(a_i)a_i'.$$

Since $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ then $\tilde{\alpha}+\tilde{\alpha}^{-1}=\tilde{\beta}+\tilde{\beta}^{-1}$ on $B(H)$. By the theorem above, it follows that $\tilde{\alpha}=\tilde{\beta}$ or $\tilde{\alpha}=\tilde{\beta}^{-1}$ and hence $\alpha=\beta$ or $\alpha=\beta^{-1}$ on M.

We conclude the note with the following

PROBLEM. Can we improve the above corollary by dropping the innerness of automorphisms?

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