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ON A COEFFICIENT PROBLEM OF ANALYTIC SYMMETRIC FUNCTIONS

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1. Introduction.

Through the paper, we consider functions f analytic in the unit disk $D=\{z : |z|<1\}$ normalized by $f(0)=0$ and $f'(0)=1$. Following Rogosinski [5], we let T denote the class of all typically-real functions f which preserve half-plane in the following sense

(1) $\qquad \operatorname{Im} z \cdot \operatorname{Im} f(z) \geqq 0 \qquad$ for $z \in D$.

Later, Robertson [4] extended the idea of Rogosinski by defining the class Q of quadrant preserving functions f which satisfy (1) and

(2) $\qquad \operatorname{Re} z \cdot \operatorname{Re} f(z) \geqq 0 \qquad$ for $z \in D$.

Recently, Y. Abu-Muhanna and T. H. MacGregor [1] have extended the quadrant preserving to sector preserving functions. Following their notations, we let A_j and B_j be the sectors defined by

$$A_j=\left\{w : \frac{2(j-1)\pi}{k}<\arg w<\frac{2j\pi}{k}\right\} \text{ and } B_j=\{e^{-i\pi/k}w : w \in A_j\},$$

where k is a positive integer and $j=1, 2, \ldots, k$. Also let $T_k, \tilde{T}_k$, and Q_k be the three classes defined respectively by

$f \in T_k$ if $f(z) \in \bar{A}_j$ whenever $z \in \bar{A}_j \cap D$, $j=1, 2, \ldots, k$,
$f \in \tilde{T}_k$ if $f(z) \in \bar{B}_j$ whenever $z \in \bar{B}_j \cap D$, $j=1, 2, \ldots, k$,
$Q_k=T_k \cap \tilde{T}_k$, where $\bar{A}$ denotes the closure of A.

Clearly, we have $T_2=T$ and $Q_2=Q$. Geometrically speaking, functions of this kind Q_k are k-fold symmetric [1, Lemma 5].

We need one more definition P_k which denotes the class of all functions p satisfying $p(0)=1$, $p(z)$ is real when z is real, $p(z)\neq 0$ and $|\arg p(z)|<\pi/k$ for $z \in D$. In [1, Theorem 5], Abu-Muhanna and MacGregor proved that a function $f \in Q_k$ if and only if there is a function $p \in P_k$

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so that

(3) $$f(z)=\frac{z}{(1-z^k)^{2/k}}p(z^k).$$

It is known that if a function f is k-fold symmetric then its power series has the form, see [1, formula (23)],

(4) $$f(z)=\sum_{n=0}^{\infty}a_{nk+1}z^{nk+1} \qquad \text{for } z\in D.$$

In [1, Theorem 7], Abu-Muhanna and MacGregor obtained the sharp bound for the first two coefficients for functions in Q_k. They then posed the question to solve the coefficient problem of functions in Q_k. The purpose of this paper is to give an answer as follows

THEOREM 1. *If $f\in Q_k$, $k\geq 2$, and is written of the form* (4), *then we have the following upper bound which is sharp when $k=2$,*

(5) $$|a_{nk+1}|\leqq\left[\left(\frac{2}{k}+1\right)\left(\frac{2}{k}+2\right)\cdots\left(\frac{2}{k}+n-1\right)/(n-1)!\right]\left(\frac{2}{nk}+\frac{4}{k}\right),$$

$n=1, 2, \ldots$.

Notice that Theorem 1 is true for $f\in T_k$ due to [1, Lemma 6]. Also notice that the problem of Abu-Muhanna and MacGregor needs to be solved only for the cases $k>2$, because when $k=2$ the solution was a consequence of Rogosinski theorem [5, p. 116], i. e.

(6) $$|a_{2n+1}|\leqq 2n+1, \qquad n=1, 2, \ldots .$$

In fact, (6) derives from $|a_{n+1}-a_{n-1}|\leqq 2$, see Pommerenke [3, p. 55]. The function

$$f(z)=\frac{z}{1-z^2}\cdot\frac{1+z^2}{1-z^2}=\sum_{n=0}^{\infty}(2n+1)z^{2n+1},$$

shows that the solution (6) is best possible.

2. Coefficient estimates of the classes P_k.

In equation (3), we observe that all coefficients in the first factor are positive and therefore all we need to do is to estimate the coefficients for functions in the classes P_k. Notice that the classes P_k are well defined for all positive real numbers.

In [2], we proved that if $p(z)=\Sigma c_n z^n$ is a function in P_k $(k\geq 2)$ then $|c_n|\leq 2\tan(\pi/k)$. We then conjectured [2, p. 142] that $|c_n|\leq 4/k$.

We now state and prove this conjecture as follows.

THEOREM 2. *If* $p(z)=1+\sum_{n=1}^{\infty}c_n z^n$ *is a function in* P_k *where* $k\geq 2$ *is real, then we have the following sharp upper bound*

$$|c_n|\leqq \frac{4}{k},\quad n=1,2,\ldots . \tag{7}$$

3. Rogosinski subordination.

To prove Theorem 2, we shall need the following subordinate principle of Rogosinski [6, Theorem X]. Recall that a function f is subordinate to another one F if there is a function g analytic and bounded by one in D such that $g(0)=0$ and $f(z)=F(g(z))$ for $z\in D$.

THEOREM R. *Let* f *and* F *be two functions analytic in* D *and normalized by* $f(0)=F(0)=0$, *but* $f'(0)=c_1$ *and* $F'(0)=C_1$. *If* f *is subordinate to* F *and if* F *is convex and univalent in* D, *then the* n*-th coefficient* c_n *of* f *satisfies*

$$|c_n|\leqq|C_1|,\qquad n=1,2,\ldots .$$

4. Proof of Theorem 2.

For convenience, we write

$$f(z)=p(z)-1\qquad\text{and}\qquad F(z)=\left(\frac{1+z}{1-z}\right)^{2/k}-1 .$$

Then by the definition of P_k, we can see that f is subordinate to F. Clearly, the function F is convex and univalent in D. It follows from Theorem R that

$$|c_n|\leqq|C_1|=\frac{4}{k},\qquad n=1,2,\ldots .$$

This proves the assertion (7).

The function

$$p_k(z)=\left(\frac{1+z}{1-z}\right)^{2/k}=1+\frac{4}{k}z+\ldots ,$$

belongs to the class P_k which shows the sharpness of (7). This completes the proof.

5. Proof of Theorem 1.

Let $f\in Q_k$, then f can be represented by equation (3) for some function $p\in P_k$. Clearly, the first factor of f can be expanded as

(8) $$\frac{z}{(1-z^k)^{2/k}}=z+\sum_{n=1}^{\infty}b_{nk+1}z^{nk+1},$$

where $$b_{nk+1}=\frac{2}{k}\left(\frac{2}{k}+1\right)\dots\left(\frac{2}{k}+n-1\right)/n!.$$

Moreover, by expanding p as a power series,

$$p(z)=1+\sum_{n=1}^{\infty}c_nz^n \qquad \text{for } z\in D,$$

we obtain the inequality (7). For convenience, we write $t=4/k$ and we define "the maximum coefficient function" of p by

(9) $$p_t(z)=1+\sum_{n=1}^{\infty}tz^n.$$

Since all coefficients of (8) and (9) are positive, it follows from (3), (4), and (7) that the coefficients a_{nk+1} of f are dominated by that of the following function

$$f_t(z)=\frac{z}{(1-z^k)^{2/k}}p_t(z^k)=\frac{z}{(1-z^k)^{2/k}}\left(1+\frac{tz^k}{1-z^k}\right)$$
$$=z+\sum_{n=1}^{\infty}\left[\left(\frac{2}{k}+1\right)\left(\frac{2}{k}+2\right)\dots\left(\frac{2}{k}+n-1\right)/(n-1)!\right]\left(\frac{2}{nk}+t\right)z^{nk+1}.$$

This concludes the assertion (5) and the proof is complete.

6. Extension of Herglotz formula.

Finally, we shall extend the representation of Herglotz formula, see [3, Theorem 2.4]. For this, we define the new class $\tilde{P}_k=\{p : p(0)=1 \text{ and } |\arg p(z)|<\pi/k\}$.

THEOREM 3. *Let $k\geqq 2$ be real and let $\mathcal{P}$ be the set of all probability measures on $[0, 2\pi]$. Then a function $p\in\tilde{P}_k$ if and only if*

(9) $$p(z)=\left\{\int_0^{2\pi}\frac{1+e^{-it}z}{1-e^{-it}z}d\mu(t)\right\}^{2/k} \text{ for some } \mu\in\mathcal{P}.$$

Moreover, the n-th coefficient c_n of p satisfies the sharp inequality

(10) $$|c_n|\leqq\frac{4}{k}, \qquad n=1, 2, \dots .$$

Proof. Clearly, $p\in\tilde{P}_k$ if and only if $p^{k/2}\in\tilde{P}_2$. Hence the assertion (9) follows from Herglotz formula. And the assertion (10) is a consequence of Theorem 2.

There is an interesting subclass of $\tilde{P}_k$ which can also be characterized by an analogue of Herglotz formula.

THEOREM 4. *Let $k \geqq 2$ be real, $\mathcal{P}$ the set of all probability measures on $[0, 2\pi]$, and P_k^* the set of all functions p_μ on D defined by*

$$(11) \qquad p_\mu(z) = \int_0^{2\pi} \left(\frac{1+e^{-it}z}{1-e^{-it}z}\right)^{2/k} d\mu(t) \qquad (\mu \in \mathcal{P}).$$

Then the class P_k^* is a proper subset of $\tilde{P}_k$ for all $k \geqq 4$ and the n-th coefficient c_n of p_μ satisfies

$$(12) \qquad |c_n| \leqq |c_n(k)| \leqq \frac{4}{k}, \qquad n=1, 2, \dots ,$$

where $c_n(k)$ is the n-th coefficient of the function $[(1+z)/(1-z)]^{2/k}$.

Proof. Clearly, the kernel functions are

$$\left(\frac{1+e^{-it}z}{1-e^{-it}z}\right)^{2/k} \in \tilde{P}_k \qquad \text{for all } t \in [0, 2\pi],$$

and the set $\{w : |\arg w| < \pi/k\}$ is convex. Since each function p_μ defined by (11) is a convex combination of those kernel functions, hence p_μ belongs to the class $\tilde{P}_k$. This shows that the class P_k^* is a subset of $\tilde{P}_k$.

To prove that the class P_k^* is a proper subset of $\tilde{P}_k$, we shall first prove the inequality (12). To do this, we need only expand the function p_μ as a power series. According to (11), it is easy to see that

$$p_\mu(z) = 1 + \sum_{n=1}^{\infty} \left[c_n(k) \int_0^{2\pi} e^{-int} d\mu(t) \right] z^n.$$

This together with Theorem 2 yields the inequality (12).

In particular, when $n=2$ we have

$$(13) \qquad c_2(k) = 8/k^2 \text{ and } c_2(4) = 1/2 \text{ for } k=4.$$

With the help of (13), we are now able to prove that the class P_k^* is a proper subset of $\tilde{P}_k$ for each $k \geqq 4$. For this, we let $p(z) = 1 + (\sin \pi/k) z^2$. By a simple computation, we find that $p \in \tilde{P}_k$ for $k \geqq 4$. Since

$$\sin \pi/k > \pi/k - \pi^3/6k^3 > 8/k^2 \text{ for } k \geqq 4,$$

it follows that $p \notin P_k^*$. This completes the proof.

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