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A NOTE ON HOLOMORPHIC VECTOR BUNDLES OVER COMPLEX TORI

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1. Let L be a lattice in C^n. A holomorphic automorphic factor of rank r for the lattice L is a holomorphic mapping

$$J : L \times C^n \longrightarrow GL(r;C)$$

such that

(1) $J(\alpha, z)$ $(\alpha \in L,\ z \in C^n)$ is holomorphic in z,

(2) $J(\alpha+\beta, z) = J(\alpha, z+\beta)J(\beta, z)$ for all $\alpha, \beta \in L$ and $z \in C^n$.

We have the following free action of L on $C^n \times C^r$ defined by

$$(z, \xi)\alpha = (z+\alpha, J(\alpha, z)\xi),\ \alpha \in L, z \in C^n,\ \ \xi \in C^r.$$

The quotient of $C^n \times C^r$ by this group action of L is a holomorphic vector bundle E_J over the complex torus $M = C^n/L$. Holomorphic vector bundles over the complex torus $M = C^n/L$ are always obtained in this way.

In this short paper, we characterize projectively flat vector bundles over a complex torus which is simple.

2. A holomorphic vector bundle over a complex manifold is said to be simple if its endomorphisms are all scalars. It is easy to show that the vector bundle E_J over the complex torus $M = C^n/L$ defined by an automorphic factor J is simple if and only if scalars are holomorphic maps $B : C^n \rightarrow GL(r;C)$ such that

$$B(z+\alpha)\ J(\alpha, z) = J(\alpha, z)B(z) \text{ for all } \alpha \in L.$$

In his paper [1], Morikawa shows the following theorem.

THEOREM. *Let J be a holomorphic automorphic factor of rank r for the lattice L in C^n such that*

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(1) *the associated vector bundle* E_J *is simple,*

(2) $J(\alpha, z+\beta)J(\alpha, z)^{-1}$ $(\alpha, \beta\in L)$ *are constants.*

Then there exists an isogeny $\phi : N\to \mathbf{C}^n/L$ *of degree* r *and a line bundle* F *over the complex torus* N *such that* E_J *is the direct image of* F *under the isogeny* ϕ.

REMARK. *We can also show that* $T_g^*F\not\cong F$ *for all* $g\in \ker \phi$, *where* T_g *is the translation of the complex torus* N *by* $g\in N$.

Let J *be an automorphic factor of rank* r *for the lattice* L *in* $\mathbf{C}^n$ *satisfying the hypotheses in the above theorem. Then, by Oda [2], the homomorphism* $H^j(M, \mathcal{O}) \to H^j(M, \mathrm{End}(E_J))$ *induced by* $\mathcal{O}\to \mathrm{End}(E_J)$ *is an isomorphism for each* j, *where* $M=\mathbf{C}^n/L$. *Therefore we obtain, for each* j,

$$\dim_{\mathbf{C}} H^j(M, \mathrm{End}(E_J)) = \dim_{\mathbf{C}} H^j(M, \mathcal{O}) = \binom{n}{j}.$$

3. Let E a holomorphic vector bundle of rank r over a complex torus $M=\mathbf{C}^n/L$ with the corresponding automorphic factor $J : L\times \mathbf{C}^n \to GL(r;\mathbf{C})$. Let

$$p : GL(r;\mathbf{C}) \to PGL(r;\mathbf{C})$$

be the natural projection to the projective general linear group and define

$$\hat{J}=p\circ J : L\times \mathbf{C}^n \to PGL(r;\mathbf{C}).$$

Then $\hat{J}$ is an automorphic factor for the projective bundle $P(E)$ over M.

We assume that E is projectively flat. Then $P(E)$ is defined by a representation of L into $PGL(r;\mathbf{C})$. Replacing J by an equivalent automorphic factor, we may assume that $\hat{J}$ is a representation of L into $PGL(r;\mathbf{C})$. Since $\hat{J}$ is independent of z, we can write

$$J(\alpha, z)=f(\alpha, z)J(\alpha, 0), \quad \alpha\in L, \quad z\in \mathbf{C}^n,$$

where $f : L\times \mathbf{C}^n\to \mathbf{C}^*$ is a scalar function. Since $\det J$ is an automorphic factor for the line bundle $\det E$, we may write

$$\det J(\alpha, z) = \chi(\alpha) \exp\left\{H(z, \alpha) + \frac{1}{2}H(\alpha, \alpha)\right\}, \alpha\in L, \quad z\in \mathbf{C}^n,$$

where $\chi : L \to \mathbf{C}^*$ is a semi-character of L, and H is an Hermitian form on $\mathbf{C}^n$. Since $f(\alpha, 0)=f(0, z)=1$,

$$f(\alpha, z) = \exp\left\{\frac{1}{r}H(z, \alpha)\right\}, \quad \alpha\in L, \quad z\in \mathbf{C}^n.$$

Now we assume that E admits a projectively flat Hermitian structure h. Then we may assume that $\hat{J}$ is a representation of L into $PU(r)$. Let $\tilde{h}$ be the induced Hermitian structure in $\tilde{E}=\pi^*E=\boldsymbol{C}^n\times\boldsymbol{C}^r$, where $\pi:\boldsymbol{C}^n\to\boldsymbol{C}^n/L$ is the natural projection. Then

(A) $\tilde{h}(z)={}^t\overline{J(\alpha,z)}\tilde{h}(z+\alpha)J(\alpha,z),\quad \alpha\in L,\quad z\in\boldsymbol{C}^n.$

The curvature form Ω of E is of the form

$$\Omega=\delta I_r,\quad \delta \text{ is a 2-form.}$$

Since its trace is the curvature of det E, we have

$$\delta=\frac{1}{r}\sum_{j,k} H_{j\bar{k}}\ dz^j\wedge d\bar{z}^k,$$

with constant coefficient $H_{j\bar{k}}$. Thus

$$\begin{aligned}\tilde{\omega}(z)&=\tilde{h}(z)^{-1}\partial\tilde{h}(z)\\&=-\frac{1}{r}H(dz,z)I_r+\Xi(z),\end{aligned}$$

where $H(dz,z)=\sum H_{j\bar{k}}\bar{z}^k dz^j$ and Ξ is a holomorphic 1-form with values in the Lie algebra of $CU(r)=\{cU;\ c\in\boldsymbol{C}^*,\ U\in U(r)\}$. Since $\Xi+{}^t\overline{\Xi}=\phi I_r$ (ϕ is a 1-form) and Ξ is holomorphic, it follows that

$$\Xi=\theta I_r,\quad \theta \text{ is a holomorphic 1-form.}$$

By a simple calculation, we obtain

$$\Xi(z+\alpha)=\Xi(z),\quad \alpha\in L.$$

That is, θ is a holomorphic 1-form on $M=\boldsymbol{C}^n/L$. Hence

$$\theta=\sum_{j=1}^{n}C_j dz_j,\quad C_j\text{'s are constants.}$$

If we solve the differential equation

$$\tilde{h}^{-1}\partial\tilde{h}=\tilde{\omega}(z)=-\frac{1}{r}\sum H_{j\bar{k}}\bar{z}^k dz^j+\sum_{j=1}^{n}C_j dz^j,$$

we obtain

$$\tilde{h}(z)=\tilde{h}(0)\ \exp\left\{-\frac{1}{r}H(z,z)+C(z)+\overline{C(z)}\right\},$$

where $C(z)=\sum_{j=1}^{n}C_j z^j$.

Using the isomorphism of the bundle $\tilde{E}$ defined by

$$(z,\xi)\in\tilde{E}\longrightarrow(z,\ \exp\ \{C(z)\}\xi)\in\tilde{E},$$

we may assume that $C(z)=0$. By a linear change of coordinates in $\boldsymbol{C}^r$, we may assume that $\tilde{h}(0)=I_r$. Therefore

$$\tilde{h}(z)=\exp\left\{-\frac{1}{r}H(z,z)\right\}I_r,\quad z\in \boldsymbol{C}^n.$$

By the formular (A)

$$J(\alpha,z)=U(\alpha)\ \exp\left\{\frac{1}{r}H(z,\alpha)+\frac{1}{2r}H(\alpha,\alpha)\right\},$$

where $U(\alpha)=J(\alpha,0)\ \exp\left\{-\frac{1}{2r}H(\alpha,\alpha)\right\}$ is a unitary matrix.

In summary, if E admits a projectively flat Hermitian structure h, its associated automorphic factor J can be written as follows:

(B) $J(\alpha,z)=U(\alpha)\ \exp\left\{\frac{1}{r}H(z,\alpha)+\frac{1}{2r}H(\alpha,\alpha)\right\},\quad \alpha\in L,\quad z\in \boldsymbol{C}^n,$

where

(i) H is an Hermitian form on $\boldsymbol{C}^n$ and its imaginary part A satisfies

$$\frac{1}{\pi}A(\alpha,\beta)\in Z \text{ for } \alpha,\beta\in L,$$

(ii) $U: L\rightarrow U(r)$ is a semi-representation in the sense that it satisfies

$$U(\alpha+\beta)=U(\alpha)U(\beta)\ \exp\left\{\frac{i}{r}A(\beta,\alpha)\right\},\quad \alpha,\beta\in L.$$

4. Let E be a simple holomorphic vector bundle of rank r over the compex torus $M=\boldsymbol{C}^n/L$ which admits a projectively flat Hermitian structure. Then its automorphic factor J is given by the formula (B). Then, for all $\alpha,\beta\in L$,

$$\begin{aligned}&J(\alpha,z+\beta)J(\alpha,z)^{-1}\\ &=U(\alpha)\ \exp\left\{\frac{1}{r}H(z+\beta,\alpha)+\frac{1}{2r}H(\alpha,\alpha)\right\}\\ &\quad U(\alpha)^{-1}\exp\left\{-\frac{1}{r}H(z,\alpha)-\frac{1}{2r}H(\alpha,\alpha)\right\}\\ &=\exp\left\{\frac{1}{r}H(\beta,\alpha)\right\}\end{aligned}$$

By Theorem (Morikawa), there exists a sublattice $\tilde{L}$ of L and a line bundle F over $N=\boldsymbol{C}^n/\tilde{L}$ such that $[L:\tilde{L}]=r$ and $\phi_*F\cong E$, where $\phi:\boldsymbol{C}^n/\tilde{L}\rightarrow \boldsymbol{C}^n/L$ is the natural isogeny. And we have

$$H^j(M,\mathcal{O})\cong H^j(M,\ \mathrm{End}(E)) \text{ for all } j$$

and

$$\dim_{\boldsymbol{C}}H^j(M,\ \mathrm{End}(E))=\binom{n}{j}.$$

We set

$$\omega_\alpha(z)=J(\alpha, z)^{-1}\ dJ(\alpha, z),\quad \alpha\in L.$$

Then we obtain a system of integrable connections satisfying

(1) $d\omega_\alpha(z)+\omega_\alpha(z)\wedge\omega_\alpha(z)=0$

(2) $\omega_{\alpha+\beta}(z)=\omega_\alpha(z)+J(\alpha, z)^{-1}\ \omega_\beta(z+\alpha)J(\alpha, z),\ \alpha, \beta\in L.$

If we write

$$\omega_\alpha(z)=\sum_{l=1}^{n}A_{\alpha l}(z)dz_l\quad (\alpha\in L,\quad 1\leqq l\leqq n),$$

then all $A_{\alpha l}(z)$ are constants. Indeed,

$$\begin{aligned}&\omega_\alpha(z+\beta)-\omega_\alpha(z)\\&=J(\alpha, z+\beta)^{-1}dJ(\alpha, z+\beta)-J(\alpha, z)^{-1}dJ(\alpha, z)\\&=J(\alpha, z+\beta)^{-1}\ d(J(\alpha, z+\beta)J(\alpha, z)^{-1})\ J(\alpha, z)\\&=0.\end{aligned}$$

Since $M=\boldsymbol{C}^n/L$ is compact and $A_{\alpha l}(z+\beta)=A_{\alpha l}(z)$ $(\beta\in L)$, all $A_{\alpha l}$ are constants.

And we have $\omega_\alpha(z)\wedge\omega_\alpha(z)=0$ and so $[A_{\alpha l}, A_{\alpha m}]=0$ for all $\alpha\in L$, $1\leqq l, m\leqq n$. If we define

$$\tilde{J}(\alpha, z)=J(\alpha, 0)\ \exp\left(\sum_{l=1}^{n}A_{\alpha l}\ z_l\right),$$

then we have

$$\tilde{J}(\alpha, z)^{-1}d\tilde{J}(\alpha, z)=J(\alpha, z)^{-1}\ dJ(\alpha, z).$$

Since $\tilde{J}(\alpha, 0)=J(\alpha, 0)$, we have $\tilde{J}(\alpha, z)=J(\alpha, z)$ $(\alpha\in L)$. It is easy to show that the total Chern class of E is given by

$$c(E)=\left(1+\frac{c_1(E)}{r}\right)^r.$$

That is, the k-th Chern class of E is given by

$$c_k(E)=\binom{r}{k}\frac{1}{r^k}c_1(E)^k.$$

It is also easy to show the following identity:

$$c_2(\mathrm{End}\ E)=-(r-1)c_1^2(E)+2rc_2(E)=0.$$

In summary, we have

THEOREM. *Let E be a simple holomorphic vector bundle of rank r over the complex torus $M=\boldsymbol{C}^n/L$. Assume E admits a projectively flat Hermitian structure. Let J be its associated automorphic factor for L given* *by the formula (B). Let* $\omega_\alpha(z)=J(\alpha,z)^{-1}dJ(\alpha,z)$ $(\alpha\in L)$ *be a system of integrable connections. Then*

(1) *There exists an isogeny* $\psi: N\to M$ *of degree r and a line bundle F such that* $\psi_*F\cong E$.

(2) $H^j(M,\mathcal{O})\cong H^j(M, \mathrm{End}(E))$ *for all j.*

(3) *All* $A_{\alpha l}(\alpha\in L,\ 1\leq l\leq n)$ *are constants and* $J(\alpha,z)=J(\alpha,0)\exp(\sum_l A_{\alpha l}z_l)$, $\alpha\in L$, $z\in \boldsymbol{C}^n$.

(4) $c(E)=\left(1+\frac{c_1(E)}{r}\right)^r$

and

$$c_2(\mathrm{End}\ E)=0.$$

References

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