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NOTES ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS INTRODUCED BY SALAGEAN

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1. Introduction

Let A be the class of functions of the form

$$f(z)=z+\sum_{j=2}^{\infty}a_j z^j \tag{1.1}$$

which are analytic in the unit disk $U=\{z:|z|<1\}$. We denote by S the subclass of A consisting of all univalent functions in the unit disk U. A function $f(z)$ belonging to A is said to be starlike of order α if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \tag{1.2}$$

for some α $(0\leqq\alpha<1)$, and for all $z\in U$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α in the unit disk U. A function $f(z)$ belonging to A is said to be convex of order α if and only if

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \tag{1.3}$$

for some α $(0\leqq\alpha<1)$, and for all $z\in U$. Also we denote by $K(\alpha)$ the class of all convex functions of order α in the unit disk U. Note that $f(z)\in K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$, and that

$$K(\alpha)\subseteq K(0)\equiv K,\quad S^*(\alpha)\subseteq S^*(0)\equiv S^*,\quad \text{and } K(\alpha)\subset S^*(\alpha)\subset S$$

for $0\leqq\alpha<1$ (cf. [8]).

The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [14], and were studied subsequently by Schild [18], MacGregor[6], Pinchuk [13], Jack [3], and others.

For a function $f(z)$ in A, we define

$$D^0f(z)=f(z), \tag{1.4}$$

$$D^1f(z)=Df(z)=zf'(z), \tag{1.5}$$

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and

(1.6) $$D^n f(z) = D(D^{n-1} f(z)) \qquad (n \in N = \{1, 2, 3, \ldots\}).$$

With the help of the symbol $D^n f(z)$, Salagean [17] introduced the subclass $S_n(\alpha)$ of A consisting of functions $f(z)$ which satisfy the condition

(1.7) $$\mathrm{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \alpha \qquad (n \in N_0 = N \cup \{0\})$$

for some α $(0 \leqq \alpha < 1)$, and for all $z \in U$.

Since

(1.8) $$\frac{D^1 f(z)}{D^0 f(z)} = \frac{z f'(z)}{f(z)}$$

and

(1.9) $$\frac{D^2 f(z)}{D^1 f(z)} = 1 + \frac{z f''(z)}{f'(z)},$$

we observe that $S_0(\alpha) = S^*(\alpha)$ and $S_1(\alpha) = K(\alpha)$.

Let $f(z)$ and $g(z)$ be analytic in the unit disk U. Then a function $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in the unit disk U satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. We denote by $f(z) \prec g(z)$ this relation. In particular, if $g(z)$ is univalent in the unit disk U the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

The concept of subordination can be traced back to Lindelöf [4], but Littlewood [5] and Rogosinski [16] introduced the term and discovered the basic relations. Recently, Suffridge [19], Hallenbeck and Ruscheweyh [2], Miller and Mocanu [8], Obradović [10], and Fukui, Sakaguchi and Owa [1] proved various results for subordinate functions.

2. Application of Robertson's Result

We begin with the statement of the following result due to Robertson [15].

LEMMA 1. *Let $f(z) \in S$. For each $0 \leqq t \leqq 1$ let $F(z, t)$ be regular in the unit disk U, let $F(z, 0) \equiv f(z)$ and $F(0, t) \equiv 0$. Let p be a positive real number for which*

(2.1) $$F(z) = \lim_{t \to +0} \frac{F(z, t) - F(z, 0)}{z t^p}$$

exists. Let $F(z,t)$ *be subordinate to* $f(z)$ *in* U *for* $0 \leqq t \leqq 1$, *then*

$$\operatorname{Re}\left\{\frac{F(z)}{f'(z)}\right\} \leqq 0 \qquad (z \in U). \tag{2.2}$$

If in addition $F(z)$ *is also regular in the unit disk* U *and* $\operatorname{Re}\{F(0)\} \neq 0$, *then*

$$\operatorname{Re}\left\{\frac{F(z)}{f'(z)}\right\} < 0 \qquad (z \in U). \tag{2.3}$$

Applying Lemma 1, we prove

THEOREM 1. *Let* $f(z) \in A$, $0 \leqq \alpha < 1$, $n \in N_0$, *and* $0 \leqq t \leqq 1$. *Further, let*

$$g(z) = \frac{1}{1-\alpha}\{D^n f(z) - \alpha D^{n-1} f(z)\} \in S \tag{2.4}$$

and

$$G(z,t) = \frac{1}{1-\alpha}\{(1-t)D^n f(z) - \alpha(1-t^2)D^{n-1} f(z)\} \prec g(z). \tag{2.5}$$

Then the function $f(z)$ *belongs to the class* $S_n(\alpha)$, *where*

$$D^{-1} f(z) = \int_0^z \frac{f(s)}{s} ds. \tag{2.6}$$

Proof. We employ the same manner as used by Obradović [10]. It is easy to see that

$$\begin{aligned} G(z) &= \lim_{t \to +0} \frac{G(z,t) - G(z,0)}{zt} \\ &= \lim_{t \to +0} \frac{\partial G(z,t)/\partial t}{z} \\ &= \frac{-D^n f(z)}{(1-\alpha)z}, \end{aligned} \tag{2.7}$$

and, that

$$g'(z) = \frac{1}{1-\alpha}\{(D^n f(z))' - \alpha(D^{n-1} f(z))'\}. \tag{2.8}$$

Furthermore, it follows from (2.7) that $\operatorname{Re}\{G(0)\} = -1/(1-\alpha) \neq 0$. Consequently, by using Lemma 1 when $p=1$, we obtain

$$\begin{aligned} \operatorname{Re}\left\{\frac{g'(z)}{G(z)}\right\} &= \operatorname{Re}\left\{\alpha \frac{z(D^{n-1} f(z))'}{D^n f(z)} - \frac{z(D^n f(z))'}{D^n f(z)}\right\} \\ &= \operatorname{Re}\left\{\alpha - \frac{D^{n+1} f(z)}{D^n f(z)}\right\} < 0, \end{aligned} \tag{2.9}$$

or

$$\text{(2.10)} \qquad \mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \alpha \qquad (z \in U).$$

This completes the assertion of Theorem 1.

Taking $n=0$ in Theorem 1, we have

COROLLARY 1 (Obradović [10]). *Let* $f(z) \in A$, $0 \leqq \alpha < 1$, *and* $0 \leqq t \leqq 1$. *Further, let*

$$\text{(2.11)} \qquad g(z) = \frac{1}{1-\alpha}\left\{f(z) - \alpha\int_0^z \frac{f(s)}{s}\,ds\right\} \in S$$

and

$$\text{(2.12)} \qquad G(z,t) = \frac{1}{1-\alpha}\left\{(1-t)f(z) - \alpha(1-t^2)\int_0^z \frac{f(s)}{s}\,ds\right\} \prec g(z).$$

Then the function $f(z)$ *belongs to the class* $S^*(\alpha)$.

Taking $n=1$ in Theorem 1, we have

COROLLARY 2 (Obradović [10]). *Let* $f(z) \in A$, $0 \leqq \alpha < 1$, *and* $0 \leqq t \leqq 1$. *Further, let*

$$\text{(2.13)} \qquad g(z) = \frac{1}{1-\alpha}\{zf'(z) - \alpha f(z)\} \in S$$

and

$$\text{(2,14)} \qquad G(z,t) = \frac{1}{1-\alpha}\{(1-t)zf'(z) - \alpha(1-t^2)f(z)\} \prec g(z).$$

Then the function $f(z)$ *belongs to the class* $K(\alpha)$.

3. Application of Miller's Result

We need the following lemma due to Miller [7] (and Miller and Mocanu [9]).

LEMMA 2. *Let* $\phi(u,v)$ *be a complex function,*

$$\phi : D \longrightarrow C,\ D \subset C \times C \quad (C \text{ is the complex plane})$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. *Suppose that* ϕ *satisfies the following conditions:*

(i) $\phi(u,v)$ *is continuous in* D;

(ii) $(1,0) \in D$ *and* $\mathrm{Re}\{\phi(1,0)\} > 0$;

(iii) $\mathrm{Re}\{\phi(iu_2, v_1)\} \leqq 0$ *for all* $(iu_2, v_1) \in D$ *and such that* $v_1 \leqq -(1+u_2^2)/2$.

Let $p(z)=1+p_1z+p_2z^2+\cdots$ *be regular in the unit disk* U, *such that* $(p(z), zp'(z)) \in D$ *for all* $z \in U$. *If*

$$\mathrm{Re}\{\phi(p(z), zp'(z))\}>0 \qquad (z\in U),$$

then $\mathrm{Re}\{p(z)\}>0$ *for* $z \in U$.

An application of Lemma 2 to the class $S_n(\alpha)$ derives

THEOREM 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S_n(\alpha)$ *with* $0 \leqq \alpha < 1$ *and* $n \in N_0$. *Then*

(3.1) $$\mathrm{Re}\left\{\left[\frac{D^nf(z)}{z}\right]^\beta\right\}>\frac{1}{2\beta(1-\alpha)+1} \qquad (z\in U),$$

where $0<2\beta(1-\alpha)\leqq 1$.

Proof. Define the function $p(z)$ by

(3.2) $$A\left[\frac{D^nf(z)}{z}\right]^\beta=p(z)+B,$$

where $A=B+1$ and $B=1/2\beta(1-\alpha)$. Then $p(z)$ is regular in the unit disk U and $p(0)=1$. Differentiating both sides of (3.2) logarithmically, we obtain

(3.3) $$\frac{z(D^nf(z))'}{D^nf(z)}=\frac{zp'(z)}{\beta\{p(z)+B\}}+1,$$

that is,

(3.4) $$\frac{D^{n+1}f(z)}{D^nf(z)}-\alpha=\frac{zp'(z)}{\beta\{p(z)+B\}}+(1-\alpha).$$

This shows from $f(z)\in S_n(\alpha)$ that

(3.5) $$\mathrm{Re}\left\{\frac{zp'(z)}{\beta\{p(z)+B\}}+(1-\alpha)\right\}>0 \qquad (z\in U).$$

Setting $p(z)=u=u_1+iu_2$ and $zp'(z)=v=v_1+iv_2$, we define the function $\phi(u, v)$ by

(3.6) $$\phi(u, v)=\frac{v}{\beta(u+B)}+(1-\alpha).$$

It follows from (3.6) that $\phi(u, v)$ is continuous in $D=(C-\{-B\})\times C$, $(1, 0)\in D$ and $\mathrm{Re}\{\phi(1, 0)\}=1-\alpha>0$, and, for all $(iu_2, v_1)\in D$ such that $v_1 \leqq -(1+u_2^2)/2$,

$$\text{(3.7)} \qquad \begin{aligned} \mathrm{Re}\{\phi(iu_2, v_1)\} &= \mathrm{Re}\left\{\frac{v_1}{\beta(iu_2+B)}\right\} + (1-\alpha) \\ &= \frac{Bv_1}{\beta(u_2^2+B^2)} + (1-\alpha) \\ &\leqq \frac{-B(1+u_2^2)}{2\beta(u_2^2+B^2)} + (1-\alpha) \\ &\leqq 0 \end{aligned}$$

provided that $0 \leqq \alpha < 1$ and $0 < 2\beta(1-\alpha) \leqq 1$. Consequently, the function $\phi(u, v)$ satisfies the conditions in Lemma 2. Thus, with the help of Lemma 2, we have

$$\mathrm{Re}\{p(z)\} > 0 \qquad (z \in U),$$

or

$$\text{(3.8)} \qquad \mathrm{Re}\left\{A\left[\frac{D^n f(z)}{z}\right]^\beta - B\right\} > 0 \qquad (z \in U).$$

This completes the proof of Theorem 2.

Letting $n=0$ in Theorem 2, we obtain

COROLLARY 3 (Owa and Obradović [12]). *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S^*(\alpha)$ *with* $0 \leqq \alpha < 1$. *Then*

$$\text{(3.9)} \qquad \mathrm{Re}\left\{\left[\frac{f(z)}{z}\right]^\beta\right\} > \frac{1}{2\beta(1-\alpha)+1} \qquad (z \in U),$$

where $0 < 2\beta(1-\alpha) \leqq 1$.

REMARK 1. Making $\alpha = 1/2$ and $\beta = 1$ in Corollary 3, we have the result by Miller and Mocanu [9].

Finally, taking $n=1$ in Theorem 2, we have

COROLLARY 4 (Owa and Obradović [12]). *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $K(\alpha)$ *with* $0 \leqq \alpha < 1$. *Then*

$$\text{(3.10)} \qquad \mathrm{Re}\{(f'(z))^\beta\} > \frac{1}{2\beta(1-\alpha)+1} \qquad (z \in U),$$

where $0 < 2\beta(1-\alpha) \leqq 1$.

REMARK 2. Making $\beta = 1/2$ in Corollary 4, we have the result by Obradović and Owa [11].

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