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ASYMPTOTIC BEHAVIOR OF GENERALIZED SOLUTIONS IN BANACH SPACES

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1. Introduction

Let X be a real Banach space with norm $|\cdot|$ and let I denote the identity operator. Then an operator $A \subset X \times X$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $|x_1 - x_2| \leqslant |x_1 - x_2 + r(y_1 - y_2)|$ for all $y_i \in Ax_i$, $i=1, 2$, and $r>0$. An accretive operator $A \subset X \times X$ is m-accretive if $R(I+rA)=X$ for all $r>0$.

We consider the initial value problem

$$\frac{du}{dt}(t) + Au(t) + G(u)(t) \ni f(t), \quad 0<t<\infty$$
$$u(0)=x, \qquad (1.1)$$

where A is an m-accretive operator in a real Banach space X, $f \in L^1(0, T; X)$, $x \in \overline{D(A)}$, and G is given mapping

$$G : C([0, \infty); \overline{D(A)}) \longrightarrow L^1(0, T; X), \text{ for } 0<T<\infty. \qquad (1.2)$$

By a recent result of Crandall and Nohel [3, Theorem 1], the problem (1.1) has a unique generalized solution $u \in C([0, \infty); X)$, provided that G satisfies a Lipschitz type condition.

In this case X is a Hilbert space, the asymptotic behavior of such a solution was studied by Aizicovici [1], Morosanu [9], Pazy [10]. The present paper is concerned with asymptotic behavior, as $t \longrightarrow \infty$, of generalized solution of (1.1) in a Banach space.

First, we prove that if $u(t)$ is a generalized solution of (1.1), then the closed convex set $\bigcap_{s \geqslant 0} \overline{co}\{u(t) : t \geqslant s\} \cap A^{-1}0$ consists of at most one point, where $\overline{co}\{u(t) : t \geqslant s\}$ is the closed convex hull of $\{u(t) : t \geqslant s\}$.

This result is applied to study the problem of weak convergence of the net $\{u(t) : t \geqslant 0\}$.

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2. Preliminaries

Let X be a real Banach space and let X^* its dual. The value of $x^* \in X^*$ at $x \in X$ will be denote by (x, x^*). With each $x \in X$, we associated the set

$$J(x) = \{x^* \in X^* : (x, x^*) = |x|^2 = |x^*|^2\}.$$

Using the Hahn–Banach theorem, it is immediately clear that $J(x) \neq \phi$ for any $x \in X$. Then multi-valued operator $J : X \longrightarrow X^*$ is called the duality mapping of X.

Let $U = \{x \in X : |x| = 1\}$ stand for the unit sphere of X. Then a Banach space X is said to be smooth provided

$$\lim_{t \to 0} \frac{|x+ty| - |x|}{t}$$

exists for each x, y in U. When this is the case, the norm of X is said to be Gâteaux differentiable. It is said to be Fréchet differentiable if for each x in U, this limit is attained uniformly for y in U. The space X is said to be uniformly Gâteaux differentiable norm if for each y in U, the limit is attained uniformly for x in U. It is also known that if X has a Fréchet differentiable norm, then J is norm to norm continuous; [2] or [5] for more details.

Let D be a subset of X. Then we denote by $\overline{D}$ the closure of D and by $\overline{co}D$ the closed convex hull of D. When $\{x_\alpha\}$ is a net in X, then $x_\alpha \longrightarrow x$ (resp. $x_\alpha \rightharpoonup x$) will denote norm (resp. weak) convergence of the net $\{x_\alpha\}$ to x.

For x and y in X, let $\langle y, x\rangle_s = \max\{(y, j) : j \in J(x)\}$. An operator $A \subset X \times X$ is accretive if and only if $\langle y_1 - y_2, x_1 - x_2\rangle_s \geqslant 0$ for all $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. Let $A \subset X \times X$ be m-accretive. Then we put

$$A^{-1}0 = \{x : x \in D(A),\ Ax \ni 0\}$$

For a function $u : [0, \infty) \longrightarrow X$, we denote by

$$\omega(u) = \{y \in X : u(t_n) \longrightarrow y \text{ for some } \{t_n\} \text{ with } t_n \longrightarrow \infty\}.$$

Consider now the initial value problem (1.1), where G satisfies (1.2), x in $\overline{D(A)}$, and $f \in L^1([0, \infty); X)$.

Recall that if $\mathcal{I}$ is an interval, then $u \in W^{1,1}(\mathcal{I}; X)$ means that there is a function $v : \mathcal{I} \longrightarrow X$ which is strongly integrable on $\mathcal{I}$. (i.e., $v \in L^1(\mathcal{I}; X)$ such that $u(t) - u(s) = \int_s^t v(\tau) d\tau$ $(t, s \in \mathcal{I})$ then $u'(t) = v(t)$

a.e. on $\mathcal{G}$).

DEFINITION 2.1. A strong solution of (1.1) on $[0,\infty)$ is a function $u\in W^{1,1}_{loc}([0,\infty);X)\cap C([0,\infty);\overline{D(A)})$, satisfiying $u(0)=x$ and $u'(t)+Au(t)+G(u)(t)\ni f(t)$, a.e. on $(0,\infty)$.

DEFINITION 2.2. A function $u\in C([0,\infty);\overline{D(A)})$ is said to be a generalized solution of equation of (1,1) if there are sequence $x_n\in\overline{D(A)}$, $f_n\in L^1_{loc}([0,\infty);X)$ and $u_n\in C([0,\infty);X)$ such that u_n is a strong solution of

$$\frac{du_n}{dt}+Au_n+G(u_n)\ni f_n,$$
$$u_n(0)=x_n,$$

$x_n\longrightarrow x$, $f_n\longrightarrow f$ on $L^1(0,T;X)$ and $u_n\longrightarrow u$ in $C([0,T];X)$, for each $0<T<\infty$.

The following result is direct consequence of [3, Theorem 1 and 2]

PROPOSION 2.1. *Let G satisfy* (1,2) *and*

$$|G(u)-G(v)|_{L^1(0,t;X)}\leqslant\int_0^t\gamma(s)\,|u-v|_{L^\infty(0,s;X)}ds,\quad 0\leqslant s\leqslant t<\infty,\quad (2.1)$$

for some $\gamma\in L^1_{loc}([0,\infty);\boldsymbol{R})$ *and every* $u,v\in C([0,\infty);\overline{D(A)})$.
For each $T\in(0,\infty)$, *there is* $\alpha_T:[0,\infty)\longrightarrow[0,\infty)$ *such that*

$$Var(G(u):[0,t])\leqslant\alpha_T(R)\,(I+Var(u:[0,t])),\quad 0\leqslant t\leqslant T\qquad(2.2)$$

and

$$|G(u)(0^+)|\leqslant\alpha_T(R),$$

whenever $u\in C([0,T];\overline{D(A)})$ *is of bounded variation and* $|u|_{L^\infty(0,T;X)}\leqslant R$. *Then, for each* $x\in\overline{D(A)}$ *and* $f\in BV_{loc}([0,\infty);X)$, *problem* (1.1) *has a unique strong solution. If* $x\in\overline{D(A)}$ *and* $f\in L^1_{loc}([0,\infty);X)$, *then* (1.1) *has a unique generalized solution.*

It is assume throughout

$$\int_s^t\langle G(v)(\tau)-G(w)(\tau),v(\tau)-w(\tau)\rangle_s d\tau\geqslant 0,$$
$$0\leqslant s\leqslant t<\infty,\quad v,w\in C([0,\infty);\overline{D(A)})\qquad(2.3)$$
$$G(v)(t)\in L^1(0,\infty;X),\ \text{for each constant function } v(t)\equiv v\in D(A)\qquad(2.4)$$

$$f \in L^1(0, \infty : X) \tag{2.5}$$
$$x \in \overline{D(A)}. \tag{2.6}$$

The following lemma is consequence of [8, Remark3. 2].

LEMMA 2. 1. *Let* $x, y \in D(A)$, $f, g \in L^1([0, \infty); X)$, *and let* u, v *be the corresponding generalized solution of* (1. 1). *If* (1. 2), (2. 1), (2, 2), *and* (2. 3) *are satisfied, then we have*

$$|u(t)-v(t)| \leqslant |u(s)-v(s)| + \int_s^t |f(\tau)-g(\tau)|\, d\tau \tag{2.7}$$

whenever $0 \leqslant s \leqslant t < \infty$.

LEMMA 2. 2 [1, Lemma 3. 1]. *Suppose that* (1. 2), (2. 1), (2. 2), (2. 3), (2. 4), *and* (2. 6) *hold. Let* u *be the generalized solution of* (1. 1) *corresponding to* $f \in L^1([0, \infty); X)$. *Then*

$$\frac{1}{2}|u(t)-y|^2 - \frac{1}{2}|u(s)-y|^2 \leqslant \int_s^t (f(\tau)-G(y)(\tau)-z, u(\tau)-y)\, d\tau \tag{2.8}$$

$$|u(t)-y| \leqslant |u(s)-y| + \int_s^t |f(\tau)-G(y)(\tau)-z|\, d\tau \tag{2.9}$$

whenever $0 \leqslant s \leqslant t < \infty$, *and* $(y, z) \in A$.

If $A^{-1}0 \neq \phi$, we can take $y \in A^{-1}0$ in (2. 9) to obtain

$$|u(t)-y| - |u(s)-y| \leqslant \int_s^t |f(\tau)-G(y)(\tau)|\, d\tau, \quad t \geqslant s \geqslant 0. \tag{2.10}$$

Consequently, $u(t)$ is bounded on $[0, \infty)$ and the function $t \longrightarrow |u(t)-y| - \int_0^t |f(\tau)-G(y)(\tau)|\, d\tau$ is nonincreasing on $[0, \infty)$. Then, since $G(v)(t) \in L^1(0, \infty; X)$, for each constant function $v(t) \equiv v \in D(A)$ and $f \in L^1(0, \infty; X)$, we deduce that

$$\lim_{t \to 0} |u(t)-y| = \rho(z)$$

exist for each $y \in A^{-1}0$.

Finally, let $\{S(t) : t \geqslant 0\}$ be a nonexpansive semigroup generated by $-A$ and let $u(t)$ be the generalized solution of (1. 1). Then we obtain that

$$|S(t)u(s) - u(t+s)| \leqslant \int_s^{s+t} |f(\tau)-G(y)(\tau)|\, d\tau \tag{2.11}$$

whenever $0 \leqslant s, t < \infty$.

From (2.11), we have

$$\lim_{s\to 0}\sup_{t>0}|S(t)u(s)-u(t+s)|=0. \tag{2.12}$$

3. Asymptotic behavior

Unless other specified, $\{S(t):t\geqslant 0\}$ denote a nonexpansive semigroup generated by $-A$.

LEMMA 3.1 *Let X be a uniformly convex Banach space and A be an m-accretive operator in $X\times X$. Let (1.2), (2.1)$\sim$(2.6) be satisfied. Let $u:[0,\infty)\longrightarrow X$ be a generalized solution of (1.1) and $A^{-1}0\neq\phi$. Let $y\in A^{-1}0$, $0<\alpha\leqslant\beta<1$ and $r=\lim_{t\to\infty}|u(t)-y|$. Then, for any $\varepsilon>0$, there is $t_0\geqslant 0$ such that*

$$|S(t)(\lambda u(s)+(1-\lambda)y)-(\lambda S(t)u(s)+(1-\lambda)y)|<\varepsilon$$

for all $s\geqslant t_0$, $t\geqslant 0$ and $\lambda\in\boldsymbol{R}$ with $\alpha\leqslant\lambda\leqslant\beta$.

Proof. Let $r>0$. Then we can choose $d>0$ so small that

$$(r+d)\left(1-c\delta\left(\frac{\varepsilon}{r+d}\right)\right)=r_0<r,$$

where δ is the modulus of convexity of norm and $c=\min\{2\lambda(1-\lambda):\alpha\leqslant\lambda\leqslant\beta\}$. Let $\varepsilon_1>0$ with $r_0+2\varepsilon_1<r$. Then we can choose $t_0\geqslant 0$ such that $|u(s)-y|\geqslant r-\varepsilon_1$ for all $s\geqslant t_0$ and $|S(t)u(s)-u(t+s)|<\varepsilon_1$ for all $t\geqslant 0$ and $s\geqslant t_0$. Suppose that

$$|S(t)(\lambda u(s)+(1-\lambda)y)-(\lambda S(t)u(s)+(1-\lambda)y)|\geqslant\varepsilon$$

for some $s\geqslant t_0$, $t\geqslant 0$ *and* $\lambda\in\boldsymbol{R}$ with $\alpha\leqslant\lambda\leqslant\beta$.

Put $u=(1-\lambda)(S(t)z-y)$ and $v=\lambda(S(t)u(s)-S(t)z)$, where $z=\lambda u(s)+(1-\lambda)y$. Then $|u|\leqslant\lambda(1-\lambda)|u(s)-y|$ and $|v|\leqslant\lambda|u(s)-z|=\lambda(1-\lambda)|y-u(s)|$. We also have that $|u-v|=|S(t)z-(\lambda S(t)u(s)+(1-\lambda)y)|\geqslant\varepsilon$ and $\lambda u+(1-\lambda)v=\lambda(1-\lambda)(S(t)u(s)-y)$. So by using the lemma in [6],

$$\begin{aligned}|\lambda(1-\lambda)(S(t)u(s)-y)|&=|\lambda u(s)+(1-\lambda)v(s)|\\&\leqslant\lambda(1-\lambda|u(s)-y|\left(1-2\lambda(1-\lambda)\delta\left(\frac{\varepsilon}{|u(s)-y|}\right)\right)\\&\leqslant\lambda(1-\lambda)(r+d)\left(1-c\delta\left(\frac{\varepsilon}{r+d}\right)\right)\\&=\lambda(1-\lambda)r_0\end{aligned}$$

and hence $|S(t)u(s)-y|\leqslant r_0$. This implies

$$|u(t+s)-y|<|S(t)u(s)-y|+\varepsilon_1\leqslant r_0+\varepsilon_1<r-\varepsilon_1.$$

On the other hand, since $|u(t+s)-y|\geqslant r-\varepsilon_1$, this is contradiction. In the case when $r=0$, for any $t, s\geqslant 0$, $y\in A^{-1}0$ and $\lambda\in \boldsymbol{R}$ with $0\leqslant\lambda\leqslant 1$,

$$\begin{aligned}
&|S(t)(\lambda u(s)+(1-\lambda)y-(\lambda S(t)u(s)+(1-\lambda)y)|\\
&\quad\leqslant\lambda|S(t)(\lambda u(s)+(1-\lambda)y)-S(t)u(s)|\\
&\qquad+(1-\lambda)|S(t)(\lambda u(s)+(1-\lambda)y-y|\\
&\quad\leqslant\lambda|\lambda u(s)+(1-\lambda)y-u(s)|+(1-\lambda)|\lambda u(s)+(1-\lambda)y-y|\\
&\quad\leqslant 2\lambda(1-\lambda)|y-u(s)|
\end{aligned}$$

So, we obtain the desired result.

Let x and y be element of a Banach space X. Then we denote by $[x, y]$ the set $\{\lambda x+(1-\lambda)y : 0\leqslant\lambda\leqslant 1\}$.

LEMMA 3.2. *Let C be a closed convex subset of a Banach space X with a Fréchet differentiable norm and $\{u_t\}$ a bounded net in C. Let $z\in\bigcap_{s\geqslant 0}\overline{co}\{u_t : t\geqslant s\}$, $y\in C$ and $\{y_t\}$ a net of element in C with $y_t\in[y, u_t]$ and $|y_t-z|=\min\{|u-z| : u\in[y, u_t]\}$. If $y_t\longrightarrow y$, then $y=z$.*

Proof. Since J is single-valued, it follows from Theorem 2.5. in [3] that $(u-y_t, J(y_t-z))\geqslant 0$ for all $u\in[y, u_t]$. Putting $u=u_t$, we have

$$(u_t-y_t, J(y_t-z))\geqslant 0. \tag{3.1}$$

Since $y_t\longrightarrow y$ and $\{u_t\}$ is bounded, there exist $K>0$ and t_0 such that $|u_{t_0}-y|\leqslant K$ and $|y_t-z|\leqslant K$ for all $t\geqslant t_0$. Let $\varepsilon\geqslant 0$ and choose $\delta>0$ so small that $2\delta K<\varepsilon$. Since the norm of X is Fréchet differentiable, we can choose $t_1\geqslant t_0$ such that $|y_t-y|\leqslant\delta$ and $|J(y_t-z)-J(y-z)|<\delta$ for all $t\geqslant t_1$. Since for $t\geqslant t_1$

$$\begin{aligned}
&|(u_t-y_t, J(y_t-z))-(u_t-y, J(y-z))|\\
&\quad=|(u_t-y_t, J(y_t-z))-(u_t-y, J(y_t-z)\\
&\qquad+(u_t-y, J((y_t-z))-(u_t-y, J(y-z))|\\
&\quad\leqslant|y_t-z||y_t-y|+|u_t-y||J(y_t-z)-J(y-z)|\\
&\quad\leqslant 2\delta K<\varepsilon
\end{aligned}$$

by using (3.1), we have

$$\begin{aligned}
(u_t-y, J(y-z))&\geqslant(u_t-y_t, J(y_t-z))-\varepsilon\\
&\geqslant 0-\varepsilon\\
&=-\varepsilon.
\end{aligned}$$

Since $z\in\bigcap_{s\geqslant 0}\overline{co}\{u_t : t\geqslant s\}$, we have $(z-y, J(y-z))\geqslant-\varepsilon$. This implies $-|z-y|^2\geqslant 0$ and hence $z=y$.

LEMMA 3.3. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, let $u(t)$ be the unique generalized solution of* (1.1), *and $A^{-1}0\neq\phi$. Let $z\in\bigcap_{s\geqslant 0}\overline{co}\{u(t):t\geqslant s\}\cap A^{-1}0$ and $y\in A^{-1}0$. Then, for any positive number ε, there is $t_0\geqslant 0$ such that*

$$(u(t)-y, J(y-z))\leqslant\varepsilon|y-z| \text{ for every } t\geqslant t_0.$$

Proof. Let $z\in\bigcap_{s\geqslant 0}\overline{co}\{u(t):t\geqslant s\}\cap A^{-1}0$, $y\in A^{-1}0$ and $\varepsilon>0$. If $y=z$, lemma 3.3 is obvious. So, let $y\neq z$. For any $t\geqslant 0$, define a unique element y_t such that $y_t\in[y,u(t)]$ and $|y_t-z|=\min\{|u-z|:u\in[y,u(t)]\}$. Then, since $y\neq z$, by lemma 3.2., we have $y_t\not\longrightarrow y$. So, we obtain $c>0$ and $\{t_n\}$ with $t_n\longrightarrow\infty$ and $|y_{t_n}-y|\geqslant c$. Setting

$$y_{t_n}=a_{t_n}u(t_n)+(1-a_{t_n})y,\quad 0\leqslant a_{t_n}\leqslant 1,$$

we also obtain $c_0>0$ so small that $a_{t_n}\geqslant c_0$ for every n. In fact since

$$c\leqslant|y_{t_n}-y|=a_{t_n}|u(t_n)-y|\leqslant a_{t_n}\left(|x-y|+\int_0^{t_n}|f(\tau)|d\tau\right)$$
$$\leqslant a_{t_n}\left(|x-y|+\int_0^{\infty}|f(\tau)|d\tau\right),$$

we may put $c_0=c/\left(|x-y|+\int_0^{\infty}|f(\tau)|d\tau\right)$. Since the limit of $|u(t)-y|$ exists, putting $k=\lim_{t\to\infty}|u(t)-y|$. We have $k>0$. If not, we have $u(t)\not\longrightarrow y$ and hence $y_t\longrightarrow y$, which contradicts $y_t\not\longrightarrow y$. Let r be a positive number such that $\varepsilon>r$ and $k>2r$. And choose $a>0$ so small that

$$(R+a)\left(1-\delta\left(\frac{c_0r}{R+a}\right)\right)<R,$$

where δ is the modulus of convexity of the norm and $R=|z-y|$. Then, by lemma 3.1., there exists $s_1\geqslant 0$ such that

$$|S(s)(c_0u(t)+(1-c_0)y)-(c_0S(s)u(t)+(1-c_0)y)|<a \tag{3.2}$$

for all $s\geqslant 0$ and $t\geqslant s_1$. We also choose $s_2\geqslant 0$ such that $|u(t)-y|\geqslant 2r$ for every $t\geqslant s_2$ and $|u(t+s)-S(t)u(s)|<r$ for every $t\geqslant 0$ and $s\geqslant s_2$. Fix t_n with $t_n\geqslant\max(s_1,s_2)$. Then since $a_{t_n}\geqslant c_0$, we have

$$c_0u(t_n)+(1-c_0)y\in[y, a_{t_n}u(t_n)+(1-a_{t_n})y]=[y, y_n].$$

Hence

$$|c_0u(t_n)+(1-c_0)y-z|\leqslant\max\{|z-y|, |z-y_{t_n}|\}=|z-y|=R.$$

By using (3.2), we obtain

$$\begin{aligned}|c_0S(s)u(t_n)+(1-c_0)y-z| &\leqslant |S(s)(c_0u(t_n)+(1-c_0)y)-z|+a\\ &\leqslant |c_0u(t_n)+(1-c_0)y-z|+a\\ &\leqslant R+a \text{ for every } s\geqslant 0.\end{aligned}$$

On the other hand, since $|y-z|=R<R+a$ and

$$\begin{aligned}|c_0S(s)u(t_n)+(1-c_0)y-y| &= c_0|S(s)u(t_n)-y|\\ &\geqslant c_0(|u(s+t_n)-y|-r)\\ &\geqslant c_0 r\end{aligned}$$

for all $s\geqslant 0$, we have, by uniform convexity,

$$\begin{aligned}\left|\frac{1}{2}((c_0S(s)u(t_n)+(1-c_0)y-z)+(y-z))\right| &< (R+a)\left(1-\delta\left(\frac{c_0r}{R+a}\right)\right)\\ &< R\end{aligned}$$

and hence

$$\left|\frac{c_0}{2}S(s)u(t_n)+\left(1-\frac{c_0}{2}\right)y-z\right| < R \text{ for all } s\geqslant 0.$$

This implies that if $u_s=\frac{c_0}{2}S(s)u(t_n)+\left(1-\frac{c_0}{2}\right)y$, then

$$|u_s+\alpha(y-u_s)-z|>|y-z| \text{ for all } \alpha\geqslant 1.$$

By Theorem 2.5., in [4], we have $(u_s+\alpha(y-u_s)-y, J(y-z))\geqslant 0$ and hence $(u_s-y, J(y-z))\leqslant 0$. Then $(S(s)u(t_n)-y, J(y-z))\leqslant 0$. Therefore

$$\begin{aligned}(u(s+t_n)-y, J(y-z)) &\leqslant |u(s+t_n)-S(s)u(t_n)|\,|y-z|\\ &\quad +(S(s)u(t_n)-y, J(y-z))\\ &<\varepsilon|y-z| \text{ for all } s\geqslant 0.\end{aligned}$$

This completes the proof.

THEOREM 3.1. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, let $u(t)$ be the unique generalized solution of* (1.1), *and $A^{-1}0\neq\phi$. Then the set $\bigcap_{s\geqslant 0}\overline{co}\{u(t) : t\geqslant s\}\cap A^{-1}0$ consists of at most one point.*

Proof. Let $y, z\in\bigcap_{s\geqslant 0}\overline{co}\{u(t) : t\geqslant s\}\cap A^{-1}0$. Then, since $\frac{y+z}{2}\in A^{-1}0$, it follows from lemma 3.3, that for any $\varepsilon>0$, there is $t_0\geqslant 0$ such that $\left(u(t+t_0)-\frac{y+z}{2}, J\left(\frac{y+z}{2}-z\right)\right)\leqslant\varepsilon\left|\frac{y+z}{2}-z\right|$ for every $t\geqslant 0$. Since $y\in\overline{co}\{u(t+t_0) : t\geqslant 0\}$, we have $\left(y-\frac{y+z}{2}, J\left(\frac{y+z}{2}-z\right)\right)\leqslant\varepsilon\left|\frac{y+z}{2}-z\right|$ and hence $(y-z, J(y-z))\leqslant 2\varepsilon|y-z|$. Then we have $|y-z|\leqslant 2\varepsilon$.

Since ε is arbitrary, we have $y=z$.

THEOREM 3.2. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, let $u(t)$ be the unique generalized solution of* (1.1), *and* $A^{-1}0\neq\phi$. *If* $\omega(u)\subset A^{-1}0$, *then the set* $\{u(t) : t\geq 0\}$ *converges weakly to some* $z\in A^{-1}0$.

Proof. Since $A^{-1}0\neq\phi$, $\{u(t) : t\geq 0\}$ is bounded. So, any sequence $\{u(t_n)\}$ with $t_n\longrightarrow\infty$ must contain a subsequence $\{u(t_{n_k})\}$ which converges weakly to some $z\in\overline{D(A)}$. Since $\omega(u)\subset A^{-1}0$ and $z\in\bigcap_{s\geq 0}\overline{co}\{u(t) : t\geq s\}$, we obtain $z\in\bigcap_{s\geq 0}\overline{co}\{u(t) : t\geq s\}\cap A^{-1}0$. Therefore, it follows from Theorem 3.1., that $\{u(t) : t\geq 0\}$ converges weakly to $z\in A^{-1}0$.

THEOREM 3.3. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm, let $u(t)$ be the unique generalized solutions of* (1.1), *and* $A^{-1}0\neq\phi$. *If* $\lim_{t\to\infty}|u(t+h)-u(t)|=0$ *for all* $h\geq 0$, *then the net* $\{u(t) : t\geq 0\}$ *converges weakly to some* $y\in A^{-1}0$.

Proof. By Theorem 3.2., it suffices to show that $\omega(u)\subset A^{-1}0$. Let $\{u(t_n)\}$ be a sequence converges weakly to some $y\in\overline{D(A)}$, as $t_n\longrightarrow\infty$. Let $\varepsilon>0$. Then $|u(t+t_n)-S(t)u(t_n)|<\varepsilon/2$ and $|u(t+t_n)-u(t_n)|<\varepsilon/2$ for all large enough n and $t\geq 0$. Since

$$|S(t)u(t_n)-u(t_n)|\leq|S(t)u(t_n)-u(t+t_n)| + |u(t+t_n)-u(t_n)|,$$

we obtain $|S(t)u(t_n)-u(t_n)|<\varepsilon$ for all large enough n and $t\geq 0$. Since $(I-S(t))$ is demiclosed [5], we have $S(t)y=y$. Thus $y\in A^{-1}0$.

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